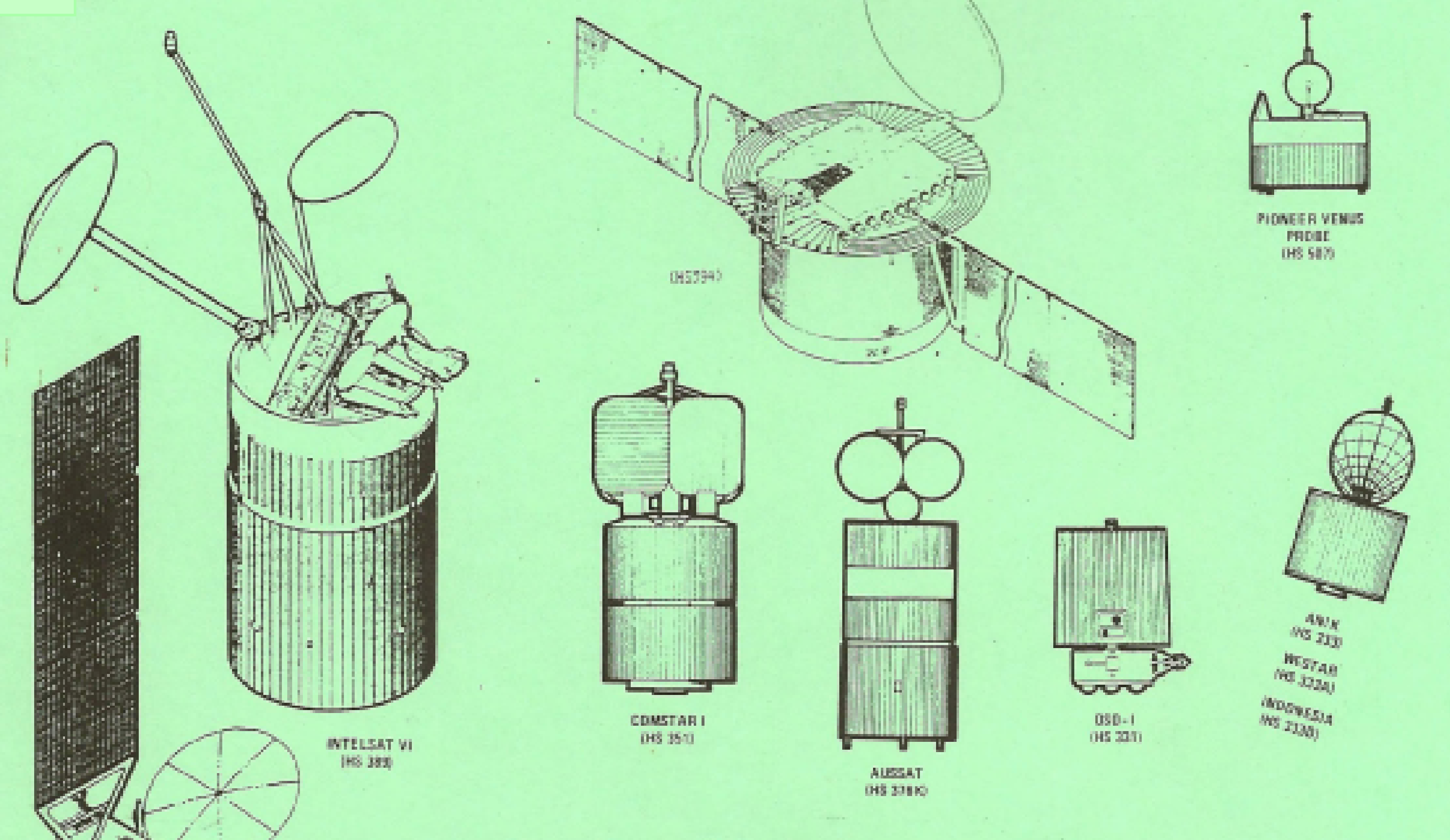
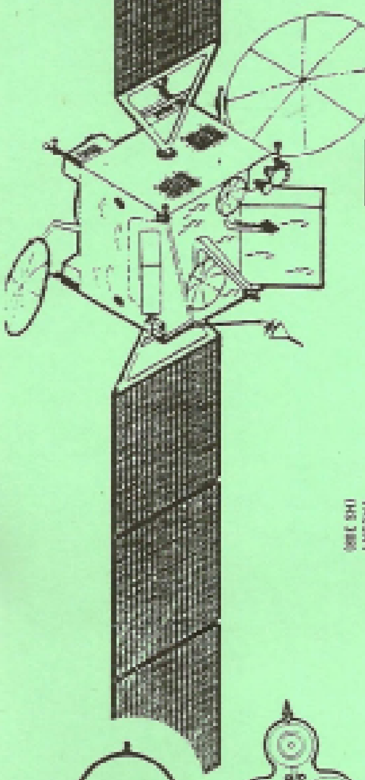
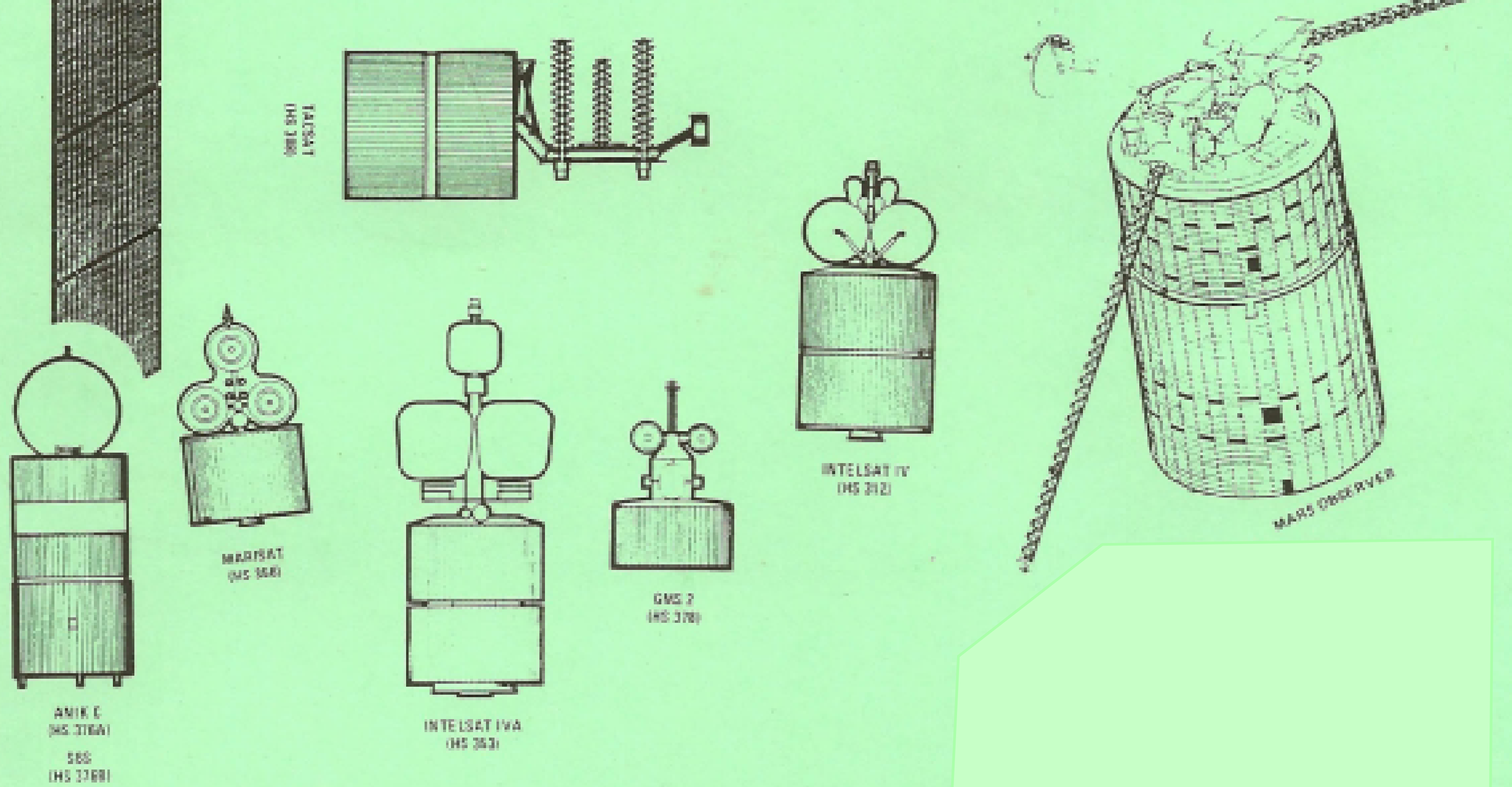




DUAL-SPIN SPACECRAFT DYNAMICS

John W. Smay



DUAL-SPIN SPACECRAFT DYNAMICS

by

John W. Smay

INTRODUCTION

Contained in the material following is a derivation and selected solutions of the rotational dynamic equations of motion for a dual-spin spacecraft and single spinning body spacecraft are covered as a subset. Since the class of body-stabilized spacecraft often have momentum bias, intentional or otherwise, much of the material applies there as well. The material has been developed, collected, and compiled by the writer over a period of several years. It arises primarily out of the writer's desire never to work the same problem twice, hence each solution is systematically recorded the third time it is obtained.

The treatment of vectors, matrices, inertia dyadics, and coordinate system (vector basis) accounting throughout is after that described in Ref. 34, Sect. 2 and/or Ref.35, Appendix B.

ACKNOWLEDGEMENT

Though most of the content is developed with the writer's thought processes and contains his biases and misconceptions, much of the material is not new or original. The Bibliography contains many of the works from which the writer first encountered some of the concepts and solutions herein, and from which the reader may seek another point of view. Over several years much of the material herein has been used, proofread, critiqued, and corrected by the writer's colleagues, the analysts in the Spacecraft Control and Dynamics design organizations at Hughes Space and Communications Company. The writer acknowledges and is grateful for this assistance.

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1.0 Momentum Derivation

The spacecraft system angular momentum with respect to an inertial point is expressed generically as

$$\mathbf{H} = \int (\mathbf{r}_o + \mathbf{r}) \times (\dot{\mathbf{r}}_o + \dot{\mathbf{r}}) dm \tag{1.1}$$

where $\mathbf{r}_o$ is the position of the vehicle mass center with respect to an inertial point and $\mathbf{r}$ is the position of the dual-spin spacecraft platform and rotor mass elements. Using definitions of $\mathbf{r}_i$, $\boldsymbol{\mu}_i$ given by Figure 1.1, the total momentum is

$$\mathbf{H} = \mathbf{H}_p + \mathbf{H}_s \tag{1.2}$$

with components

$$\begin{aligned} \mathbf{H}_p &= \int_{\mathbf{P}} (\mathbf{r}_o + \mathbf{r}_p + \boldsymbol{\mu}_p) \times (\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_p + \dot{\boldsymbol{\mu}}_p) dm \\ &= m_p (\mathbf{r}_o + \mathbf{r}_p) \times (\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_p) + \int \boldsymbol{\mu}_p \times \dot{\boldsymbol{\mu}}_p dm \qquad (1.3a) \\ &= m_p (\mathbf{r}_o \times \dot{\mathbf{r}}_o + \mathbf{r}_o \times \dot{\mathbf{r}}_p + \mathbf{r}_p \times \dot{\mathbf{r}}_o) + m_p (\mathbf{r}_p \times \dot{\mathbf{r}}_p) + \int \boldsymbol{\mu}_p \times \dot{\boldsymbol{\mu}}_p dm \ , \end{aligned}$$

and

$$\begin{aligned} \mathbf{H}_s &= \int_{\mathbf{R}} (\mathbf{r}_o + \mathbf{r}_s + \boldsymbol{\mu}_s) \times (\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_s + \dot{\boldsymbol{\mu}}_s) dm \\ &= m_s (\mathbf{r}_o + \mathbf{r}_s) \times (\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_s) + \int \boldsymbol{\mu}_s \times \dot{\boldsymbol{\mu}}_s dm \qquad (1.3b) \\ &= m_s (\mathbf{r}_o \times \dot{\mathbf{r}}_o + \mathbf{r}_o \times \dot{\mathbf{r}}_s + \mathbf{r}_s \times \dot{\mathbf{r}}_o) + m_s (\mathbf{r}_s \times \dot{\mathbf{r}}_s) + \int \boldsymbol{\mu}_s \times \dot{\boldsymbol{\mu}}_s dm \ . \end{aligned}$$

The sub-s and sub-p are used hereafter to denote properties of the rotor and platform respectively. The vectors $\boldsymbol{\mu}_i$ are assumed fixed in the respective bodies and the center of mass definition, $\int \boldsymbol{\mu}_s dm = 0$, has been used repeatedly in going from the first to the second form above. We shall immediately simplify by assuming both bodies statically balanced, mass centers on the common bearing axis, and denote the resultant restricted body cm positions as $\mathbf{r}_s = \mathbf{r}_1$ and $\mathbf{r}_p = \mathbf{r}_2$. The result of this is that both body cm position vectors are fixed in the respective bodies.

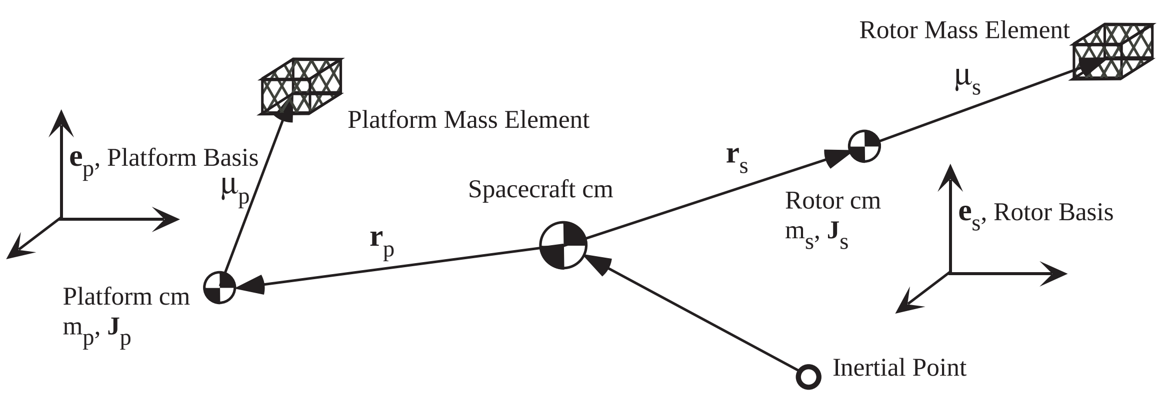

Figure 1.1 Dual-Spin Spacecraft Mass Model.

The inertial angular rates of the bodies are denoted $\boldsymbol{\omega}_i$. Then, computing the inertial time derivative

$$\dot{\boldsymbol{\mu}}_s = \frac{{}^s d\boldsymbol{\mu}_s}{dt} + \boldsymbol{\omega}_s \times \boldsymbol{\mu}_s = \boldsymbol{\omega}_s \times \boldsymbol{\mu}_s \tag{1.4}$$

$$\dot{\mathbf{r}}_1 = \frac{{}^s d\mathbf{r}_1}{dt} + \boldsymbol{\omega}_s \times \mathbf{r}_1 = \boldsymbol{\omega}_s \times \mathbf{r}_1 \tag{1.5}$$

where $\frac{{}^s d\mathbf{v}}{dt}$ indicates differentiation in a rotor fixed basis. The rotor momentum can then be expressed as

$$\begin{aligned}
\mathbf{H}_s &= m_s(\mathbf{r}_o \times \dot{\mathbf{r}}_o + \mathbf{r}_o \times \dot{\mathbf{r}}_1 + \mathbf{r}_1 \times \dot{\mathbf{r}}_o) + m_s(\mathbf{r}_1 \times \dot{\mathbf{r}}_1) + \int \boldsymbol{\mu}_s \times [\boldsymbol{\omega}_s \times \boldsymbol{\mu}_s] dm \\
&= m_s(\mathbf{r}_o \times \dot{\mathbf{r}}_o + \mathbf{r}_o \times \dot{\mathbf{r}}_1 + \mathbf{r}_1 \times \dot{\mathbf{r}}_o) + m_s(\mathbf{r}_1 \times [\boldsymbol{\omega}_s \times r_1]) + \mathbf{J}_s \cdot \boldsymbol{\omega}_s. \qquad (1.6) \\
&= m_s(\mathbf{r}_o \times \dot{\mathbf{r}}_o + \mathbf{r}_o \times \dot{\mathbf{r}}_1 + \mathbf{r}_1 \times \dot{\mathbf{r}}_o) + \mathbf{I}_s \cdot \boldsymbol{\omega}_s.
\end{aligned}$$

In the preceding $\mathbf{J}_i$ denotes the inertia dyadic of a body with respect to the body cm, while $\mathbf{I}_i$ denotes the inertia dyadic with respect to the vehicle cm. We shall attempt to hold to this convention in the following text. The inertia dyadic is introduced to represent the integral term as follows:

$$\begin{aligned}
\int \boldsymbol{\mu}_s \times [\boldsymbol{\omega}_s \times \boldsymbol{\mu}_s] dm &= \int [(\boldsymbol{\mu}_s \cdot \boldsymbol{\mu}_s)\boldsymbol{\omega}_s - (\boldsymbol{\mu}_s \cdot \boldsymbol{\omega}_s)\boldsymbol{\mu}_s] dm = \int [(\boldsymbol{\mu}_s \cdot \boldsymbol{\mu}_s)\boldsymbol{\omega}_s - \boldsymbol{\mu}_s(\boldsymbol{\mu}_s \cdot \boldsymbol{\omega}_s)] dm \qquad (1.7) \\
&= \int [(\boldsymbol{\mu}_s \cdot \boldsymbol{\mu}_s)\mathbf{U} \cdot \boldsymbol{\omega}_s - \boldsymbol{\mu}_s(\boldsymbol{\mu}_s \cdot \boldsymbol{\omega}_s)] dm = \left\{ \int [(\boldsymbol{\mu}_s \cdot \boldsymbol{\mu}_s)\mathbf{U} - \boldsymbol{\mu}_s \boldsymbol{\mu}_s] dm \right\} \cdot \boldsymbol{\omega}_s .
\end{aligned}$$

$\mathbf{U}$ is the unit dyadic defined to facilitate factoring $\boldsymbol{\omega}_s$ from the integral. In this writer's experience it is almost never necessary to expand the details of the dyadic(see Ref. 1, p. 419), however the dyadic notation is extremely useful in analysis.

Replacing $\mathbf{r}_p$ with $\mathbf{r}_2$ in Eq. 1.3a and carrying out the same manipulation yields the companion platform momentum expression,

$$\begin{aligned}
\mathbf{H}_p &= m_p(\mathbf{r}_o \times \dot{\mathbf{r}}_o + \mathbf{r}_o \times \dot{\mathbf{r}}_2 + \mathbf{r}_2 \times \dot{\mathbf{r}}_o) + m_p(\mathbf{r}_2 \times [\boldsymbol{\omega}_p \times r_2]) + \mathbf{J}_p \cdot \boldsymbol{\omega}_p. \\
&= m_p(\mathbf{r}_o \times \dot{\mathbf{r}}_o + \mathbf{r}_o \times \dot{\mathbf{r}}_2 + \mathbf{r}_2 \times \dot{\mathbf{r}}_o) + \mathbf{I}_p \cdot \boldsymbol{\omega}_p. \qquad (1.8)
\end{aligned}$$

Next we choose appropriate vector bases and define some required vector components. We define a vector basis in the form

$$\mathbf{e}_s^T = [\mathbf{e}_{s1}, \mathbf{e}_{s2}, \mathbf{e}_{s3}] , \tag{1.9}$$

where the elements are unit vectors along the three right handed orthogonal coordinates of basis $\mathbf{e}_s$, similar to the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ triad once frequently used. With this notation a vector $\mathbf{v}$ in $\mathbf{e}_s$ is written

$$\begin{aligned}
\mathbf{v} &= \mathbf{e}_s^T \, v = \mathbf{e}_s^T [v_1, v_2, v_3]^T \\
&= [\mathbf{e}_{s1}, \mathbf{e}_{s2}, \mathbf{e}_{s3}] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \mathbf{e}_{s1} + v_2 \mathbf{e}_{s2} + v_3 \mathbf{e}_{s3}.
\end{aligned}$$

As with the dyadic, we will *never* find it necessary to expand the basis as is done here to clarify the meaning.

Now we choose a rotor fixed basis $\mathbf{e}_s$ with 3-axis along the spin axis, positive toward the platform, and the remaining two axes forming a right handed triad. Further, a platform fixed basis is chosen related to the rotor basis as

$$\mathbf{e}_s = B(\psi)\mathbf{e}_p = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_p . \tag{1.10}$$

With the notation just set up the basis that the scalar components of a vector are expressed in are always evident, for example $\mathbf{v} = \mathbf{e}_s^T v$ immediately means the components of v are expressed in $\mathbf{e}_s$. Also, it is systematic and straightforward to transform from one basis to another. Again for example, transposing Eq. 1.10, $\mathbf{e}_s^T = \mathbf{e}_p^T B(\psi)^T$ and this can be substituted directly to obtain $\mathbf{v} = \mathbf{e}_s^T v = \mathbf{e}_p^T B(\psi)^T v$ so that the elements of $\mathbf{v}$ in $\mathbf{e}_p$ are immediately evident as $B(\psi)^T v$. Finally, we claim that inertia matrices, dyadics, and vector bases are handled systematically with this notation. To demonstrate, let J_s be the rotor inertia matrix(Ref. 1, p. 420), and $\mathbf{J}_s$ be the corresponding inertia dyadic. Then,

$$\mathbf{J}_s = \mathbf{e}_s^T J_s \mathbf{e}_s,$$

and if one wishes to express this inertia in the platform basis(rotate the matrix to a new coordinate system), it is again straightforward to substitute for $\mathbf{e}_s$ from 1.10 getting

$$\mathbf{J}_s = \mathbf{e}_p^T B(\psi)^T J_s B(\psi) \mathbf{e}_p,$$

so that $B(\psi)^T J_s B(\psi)$ is the rotor inertia matrix expressed in the platform basis. Next we introduce the matrix representation of the vector cross product as

$$\mathbf{u} \times \mathbf{v} = \mathbf{e}^T u \times \mathbf{e}^T v = \mathbf{e}^T \tilde{u} v = -\,\mathbf{e}^T \tilde{v} u \tag{1.11}$$

$$\tilde{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}; \;\; [\tilde{u} + \tilde{v}] = [u + v]^{\sim}\; ; \;\; \tilde{u}\tilde{v} = [\tilde{v}\tilde{u}]^T \neq \tilde{v}\tilde{u}\ ,$$

and remark for future reference that $B(\psi)\tilde{u} \neq [B(\psi)u]^{\sim}$.

Referring to 1.10 again, we denote the relative angular velocity between rotor and platform as

$$\boldsymbol{\omega}_r = \boldsymbol{\omega}_s - \boldsymbol{\omega}_p = \mathbf{e}_s^T[\,0,\,0,\,\omega_r]^T = \mathbf{e}_s^T[\,0,\,0,\,\dot{\psi}]^T. \tag{1.12}$$

It is noteworthy that a large fraction of the analysis of any given problem can be done without ever choosing the vector bases. Only when one wishes quantitatively to fix the components or impose certain system constraints need the bases be chosen. We have chosen the 3-axis of rotor and platform bases in our dual-spin vehicle by virtue of (1.10) to coincide with the bearing axis constraint and used this in (1.12). However, the 1 and 2-axes in both bodies remain arbitrary for the present. Now various previously defined vectors are assigned in their respective coordinate bases as follows:

$$\mathbf{H}_s = \mathbf{e}_s^T H_s \tag{1.13a}$$

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T \omega_s \tag{1.13b}$$

$$\mathbf{r}_s = \mathbf{e}_s^T r_s \tag{1.13c}$$

$$\mathbf{J}_s = \mathbf{e}_s^T J_s \mathbf{e}_s \tag{1.13d}$$

$$\mathbf{I}_s = \mathbf{e}_s^T I_s \mathbf{e}_s \tag{1.13e}$$

and similarly for the platform

$$\mathbf{H}_p = \mathbf{e}_p^T H_p \tag{1.14a}$$

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T \omega_p \tag{1.14b}$$

$$\mathbf{r}_p = \mathbf{e}_p^T r_p \tag{1.14c}$$

$$\mathbf{J}_p = \mathbf{e}_p^T J_p \mathbf{e}_p \tag{1.14d}$$

$$\mathbf{I}_p = \mathbf{e}_p^T I_p \mathbf{e}_p\ . \tag{1.14e}$$

Using the above definitions and taking $\mathbf{r}_o = 0$, Eq. 1.6 is rewritten

$$\mathbf{H}_s = \mathbf{e}_s^T H_s = \mathbf{e}_s^T J_s \mathbf{e}_s \cdot \mathbf{e}_s^T \omega_s - m_s \mathbf{e}_s^T r_1 \times [\mathbf{e}_s^T r_1 \times \mathbf{e}_s^T \omega_s] \tag{1.15}$$

$$= \mathbf{e}_s^T [J_s \omega_s - m_s \tilde{r}_1 \tilde{r}_1 \omega_s] = \mathbf{e}_s^T [J_s - m_s \tilde{r}_1 \tilde{r}_1] \omega_s = \mathbf{e}_s^T I_s \omega_s$$

from which we infer

$$I_s = J_s - m_s \tilde{r}_1 \tilde{r}_1 \ , \tag{1.16}$$

and

$$H_s = I_s \omega_s = [J_s - m_s \tilde{r}_1 \tilde{r}_1] \omega_s \tag{1.17}$$

gives the components of rotor momentum H_s in the rotor fixed frame $\mathbf{e}_s$. In similar fashion

$$I_p = J_p - m_p \tilde{r}_2 \tilde{r}_2 \ , \tag{1.18}$$

and

$$H_p = I_p \omega_p \ . \tag{1.19}$$

The total angular momentum with respect to the vehicle cm is

$$\mathbf{H} = \mathbf{H}_s + \mathbf{H}_p = \mathbf{I}_s \cdot \boldsymbol{\omega}_s + \mathbf{I}_p \cdot \boldsymbol{\omega}_p = \mathbf{e}_s^T I_s \omega_s + \mathbf{e}_p^T I_p \omega_p$$

$$= \mathbf{e}_p^T [B^T I_s \omega_s + I_p \omega_p] = \mathbf{e}_s^T [I_s \omega_s + B I_p \omega_p] \ . \tag{1.20}$$

Noting that

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T \omega_s = \boldsymbol{\omega}_p + \boldsymbol{\omega}_r = \mathbf{e}_p^T \omega_p + \mathbf{e}_s^T \omega_r = \mathbf{e}_s^T [B \omega_p + \omega_r] \ , \tag{1.21}$$

where the liberty is taken(see Eq. 1.12) to let the symbol ω_r represent the three-vector $\omega_r = [0, 0, \omega_r]^T$ and its 3-axis component. Similarly,

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T \omega_p = \boldsymbol{\omega}_s - \boldsymbol{\omega}_r = \mathbf{e}_s^T [\omega_s - \omega_r] = \mathbf{e}_p^T [B^T \omega_s - \omega_r] \ . \tag{1.22}$$

We have freely used the special nature of $\omega_r = B^T \omega_r = B \omega_r$ in the preceding two equations. Next, using Equation 1.21 in 1.20[1],

$$\mathbf{H} = \mathbf{e}_s^T [I_s B \omega_p + I_s \omega_r + B I_p \omega_p] = \mathbf{e}_p^T [B^T I_s B \omega_p + B^T I_s \omega_r + I_p \omega_p]$$

$$= \mathbf{e}_p^T [\{I_p + B^T I_s B\} \omega_p + B^T I_s \omega_r] = \mathbf{e}_p^T H^p \ , \tag{1.23}$$

giving an expression of total vehicle angular momentum in terms of platform and relative rate in either coordinate system. Using Equation 1.22 in 1.20 gives the parallel case in terms of ω_s and ω_r as

$$\mathbf{H} = \mathbf{e}_p^T [B^T I_s \omega_s - I_p \omega_r + I_p B^T \omega_s] = \mathbf{e}_s^T [I_s \omega_s - B I_p \omega_r + B I_p B^T \omega_s]$$

$$= \mathbf{e}_s^T [\{I_s + B I_p B^T\} \omega_s - B I_p \omega_r] = \mathbf{e}_s^T H^s \ . \tag{1.24}$$

The inertia matrix is developed in terms of basic scalar integrals in Ref. 1, p 417. Herein, we denote the elements of an inertia matrix as

$$I_i = \begin{bmatrix} I_{11}^i & -I_{12}^i & -I_{13}^i \\ -I_{12}^i & I_{22}^i & -I_{23}^i \\ -I_{13}^i & -I_{23}^i & I_{33}^i \end{bmatrix} , \tag{1.25}$$

where i = s, or p respectively for rotor or platform. At this juncture it is appropriate to dwell on the conventions for the elements of the inertia matrix. The diagonal elements $I_{ii} = \int x_i y_j dm$ are always positive so give rise to no confusion. For off diagonal elements we have inserted a negative sign in (1.25) and when scalar expansions of equations are carried out this sign is retained. This means that when substituting for I_{ij} in a scalar expansion in this document(and in most cases around Hughes) one should substitute $\int x_i y_j dm$. At this writing(January 1988) this is the

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[1] Note the subscript on H_p denotes the platform momentum only, while the superscript on H^p indicates the components of total vehicle angular momentum in platform basis $\mathbf{e}_p$.

number consistently reported by Hughes mass properties for product of inertia. Now, when the numbers are inserted in a matrix and used as in Eq. 1.24, say in computer matrix manipulation, the negative sign must be overtly inserted, i.e., the off diagonal elements in I_s, I_p of (1.24) must be $-\int x_i y_j dm$. To repeat, the off diagonal numbers supplied by mass properties must be negated before substitution in a matrix, but may be used directly in scalar expansions herein.

Before proceeding to the torque equation, we digress to note some convenient properties of H under certain mass property constraints on the rotor and/or platform. The term $B^T I_s B$ multiplying ω_p in the second form of Eq. 1.23 is the time-varying rotor inertia seen in platform coordinates, which expands as

$$B^T I_s B = \begin{bmatrix} \bar{I}^s_{11} & -\bar{I}^s_{12} & -\bar{I}^s_{13} \\ -\bar{I}^s_{12} & \bar{I}^s_{22} & -\bar{I}^s_{23} \\ -\bar{I}^s_{13} & -\bar{I}^s_{23} & \bar{I}^s_{33} \end{bmatrix} \tag{1.26}$$

$$= \begin{bmatrix} I^s_{11} + \Delta I_s \sin^2\psi + I^s_{12}\sin 2\psi & ------------ & -- \\ -I^s_{12}\cos 2\psi - (\Delta I_s/2)\sin 2\psi & I^s_{22} - \Delta I_s \sin^2\psi - I^s_{12}\sin 2\psi & -- \\ -I^s_{13}\cos\psi + I^s_{23}\sin\psi & -I^s_{23}\cos\psi - I^s_{13}\sin\psi & I^s_{33} \end{bmatrix}.$$

Equation 1.26 shows that if the rotor is symmetric (transverse inertias equal implying $\Delta I_s = I^s_{12} = 0$) and is dynamically balanced (products of inertia vanish, $I^s_{13} = I^s_{23} = 0$), then $B^T I_s B = I_s$. Also, dynamic balance alone is sufficient to render $B^T I_s \omega_r = I_s \omega_r$. Thus, for the but important special case

$$\mathbf{H} = \mathbf{e}_p^T[I_s(\omega_p + \omega_r) + I_p\omega_p] = \mathbf{e}_p^T[(I_s + I_p)\omega_p + I_s\omega_r] \text{ ; (symmetric and balanced rotor) .} \tag{1.27}$$

Similarly, with the same constraint imposed on the platform, the momentum in rotor coordinates reduces to

$$\mathbf{H} = \mathbf{e}_s^T[I_s\omega_s + I_p(\omega_s - \omega_r)] = \mathbf{e}_s^T[(I_s + I_p)\omega_s - I_p\omega_r] \text{ ; (symmetric and balanced platform) .} \tag{1.28}$$

Expanding the total system angular momentum in platform coordinates in terms of ω_p and ω_r

$$H^p = \begin{bmatrix} [I^p_{11} + \bar{I}^s_{11}]\omega_{p1} + [-I^p_{12} - \bar{I}^s_{12}]\omega_{p2} + [-I^p_{13} - \bar{I}^s_{13}]\omega_{p3} - \bar{I}^s_{13}\omega_r \\ [-I^p_{12} - \bar{I}^s_{12}]\omega_{p1} + [I^p_{22} + \bar{I}^s_{22}]\omega_{p2} + [-I^p_{23} - \bar{I}^s_{23}]\omega_{p3} - \bar{I}^s_{23}\omega_r \\ [-I^p_{13} - \bar{I}^s_{13}]\omega_{p1} + [-I^p_{23} - \bar{I}^s_{23}]\omega_{p2} + [I^p_{33} + I^s_{33}]\omega_{p3} + I^s_{33}\omega_r \end{bmatrix} \tag{1.29}$$

$$= \begin{bmatrix} [I_{11} + \Delta I_s \sin^2\psi + I^s_{12}\sin 2\psi]\omega_{p1} + [-I^p_{12} - I^s_{12}\cos 2\psi - \dfrac{\Delta I_s}{2}\sin 2\psi]\omega_{p2} - [I^s_{13}\cos\psi - I^s_{23}\sin\psi][\omega_{p3} + \omega_r] - I^p_{13}\omega_{p3} \\ [-I^p_{12} - I^s_{12}\cos 2\psi - \dfrac{\Delta I_s}{2}\sin 2\psi]\omega_{p1} + [I_{22} - \Delta I_s \sin^2\psi - I^s_{12}\sin 2\psi]\omega_{p2} - [I^s_{23}\cos\psi + I^s_{13}\sin\psi][\omega_{p3} + \omega_r] - I^p_{23}\omega_{p3} \\ [-I^p_{13} - I^s_{13}\cos\psi + I^s_{23}\sin\psi]\omega_{p1} + [-I^p_{23} - I^s_{23}\cos\psi - I^s_{13}\sin\psi]\omega_{p2} + I^s_{33}[\omega_{p3} + \omega_r] + I^p_{33}\omega_{p3} \end{bmatrix}$$

where

$$I_{ii} = I^p_{ii} + I^s_{ii} \tag{1.30a}$$

$$\Delta I_j = I^j_{22} - I^j_{11} \text{ .} \tag{1.30b}$$

The parallel expansion in rotor coordinates is obtained by exchanging I_p and I_s, replacing ω_p with ω_s, and replacing ω_r with $-\omega_r$ in Equation 1.29.

2.0 Dual-Spin Torque Equations

The torque equation is derived by first differentiating the first form in Eq. 1.20.

$$\mathbf{T} = \dot{\mathbf{H}} = \dot{\mathbf{I}}_s \cdot \boldsymbol{\omega}_s + \mathbf{I}_s \cdot \dot{\boldsymbol{\omega}}_s + \dot{\mathbf{I}}_p \cdot \boldsymbol{\omega}_p + \mathbf{I}_p \cdot \dot{\boldsymbol{\omega}}_p \ . \tag{2.1}$$

The dyadic derivatives expand as

$$\dot{\mathbf{I}}_s = \frac{{}^s d\mathbf{I}_s}{dt} + \boldsymbol{\omega}_s \times \mathbf{I}_s - \mathbf{I}_s \times \boldsymbol{\omega}_s = \boldsymbol{\omega}_s \times \mathbf{I}_s - \mathbf{I}_s \times \boldsymbol{\omega}_s \tag{2.2}$$

where ${}^s d/dt$ indicates differentiation in the $\mathbf{e}_s$ frame and the first term above vanishes because I_s is constant in this frame. In the same fashion $\dot{\mathbf{I}}_p = \boldsymbol{\omega}_p \times \mathbf{I}_p - \mathbf{I}_p \times \boldsymbol{\omega}_p$. When substituted in Equation 2.1, both $\mathbf{I}_s \times \boldsymbol{\omega}_s \cdot \boldsymbol{\omega}_s$ and $\mathbf{I}_p \times \boldsymbol{\omega}_p \cdot \boldsymbol{\omega}_p$ vanish because they are inner products of orthogonal vectors. Hence,

$$\dot{\mathbf{H}} = \boldsymbol{\omega}_s \times \mathbf{I}_s \cdot \boldsymbol{\omega}_s + \mathbf{I}_s \cdot \dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_p \times \mathbf{I}_p \cdot \boldsymbol{\omega}_p + \mathbf{I}_p \cdot \dot{\boldsymbol{\omega}}_p \ . \tag{2.3}$$

Now we want the matrix formulation of $\dot{\mathbf{H}}$. First it is obtained in terms of ω_p and ω_r , using

$$\boldsymbol{\omega}_s = \boldsymbol{\omega}_p + \boldsymbol{\omega}_r = \mathbf{e}_p^T\omega_p + \mathbf{e}_s^T\omega_r = \mathbf{e}_s^T[B\omega_p + \omega_r] \ , \tag{2.4}$$

and

$$\dot{\boldsymbol{\omega}}_s = \frac{{}^s d\boldsymbol{\omega}_s}{dt} + \boldsymbol{\omega}_s \times \boldsymbol{\omega}_s = \frac{{}^s d\boldsymbol{\omega}_s}{dt} = \mathbf{e}_s^T\dot{\omega}_s = \mathbf{e}_s^T[\dot{B}\omega_p + B\dot{\omega}_p + \dot{\omega}_r] \ . \tag{2.5}$$

Equation 2.3 becomes

$$\begin{aligned}\dot{\mathbf{H}} &= \mathbf{e}_s^T\{[\widetilde{B\omega_p} + \tilde{\omega}_r]I_s[B\omega_p + \omega_r] + I_s[\dot{B}\omega_p + B\dot{\omega}_p + \dot{\omega}_r]\} + \mathbf{e}_p^T[\tilde{\omega}_p I_p \omega_p + I_p\dot{\omega}_p] \\ &= \mathbf{e}_p^T\{B^T[\widetilde{B\omega_p} + \tilde{\omega}_r]I_s[B\omega_p + \omega_r] + B^T I_s[\dot{B}\omega_p + B\dot{\omega}_p + \dot{\omega}_r] + \tilde{\omega}_p I_p \omega_p + I_p\dot{\omega}_p\} = \mathbf{e}_p^T\dot{H}^p \\ &= \mathbf{e}_p^T\{\tilde{\omega}_p H^p + B^T\tilde{\omega}_r I_s[B\omega_p + \omega_r] + B^T I_s[\dot{B}\omega_p + B\dot{\omega}_p + \dot{\omega}_r] + I_p\dot{\omega}_p]\end{aligned} \tag{2.6}$$

where the last form with H^p is obtained using the kinematic identity $\widetilde{B\omega_p} = B\tilde{\omega}_p B^T$.

Now apply torques in the notation

$$\dot{\mathbf{H}} = \mathbf{T}_s + \mathbf{T}_p = \mathbf{e}_s^T T_s + \mathbf{e}_p^T T_p \tag{2.7}$$

such that

$$\mathbf{e}_p^T\dot{H}^p = \mathbf{e}_p^T[B^T T_s + T_p] \ . \tag{2.8}$$

Substituting Eq. 2.8 in 2.6, premultiplying both sides by $\mathbf{e}_p$, and solving for $\dot{\omega}_p$, the result is

$$\begin{aligned}\dot{\omega}_p &= [I_p + B^T I_s B]^{-1}\{-B^T[\widetilde{B\omega_p} + \tilde{\omega}_r]I_s[B\omega_p + \omega_r] - B^T I_s[\dot{B}\omega_p + \dot{\omega}_r] - \tilde{\omega}_p I_p \omega_p + B^T T_s + T_p\} \\ &= [I_p + B^T I_s B]^{-1}\{-\tilde{\omega}_p H^p - B^T\tilde{\omega}_r I_s[B\omega_p + \omega_r] - B^T I_s[\dot{B}\omega_p + \dot{\omega}_r] + B^T T_s + T_p\} \ .\end{aligned} \tag{2.9}$$

Repeating the derivation of Equations 2.4 to 2.9 in the rotor basis and expressing in terms of ω_s and ω_r ,

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T\omega_p = \mathbf{e}_p^T[B^T\omega_s - \omega_r] \ , \tag{2.10}$$

$$\dot{\boldsymbol{\omega}}_p = \frac{{}^p d\boldsymbol{\omega}_p}{dt} = \mathbf{e}_p^T\dot{\omega}_p = \mathbf{e}_p^T[\dot{B}^T\omega_s + B^T\dot{\omega}_s - \dot{\omega}_r] \ , \tag{2.11}$$

$$\begin{aligned}\dot{\mathbf{H}} &= \mathbf{e}_s^T\{\tilde{\omega}_s I_s \omega_s + I_s\dot{\omega}_s + B[\widetilde{(B^T\omega_s)} - \tilde{\omega}_r]I_p[B^T\omega_s - \omega_r] + BI_p[\dot{B}^T\omega_s + B^T\dot{\omega}_s - \dot{\omega}_r]\} = \mathbf{e}_s^T\dot{H}^s \\ &= \mathbf{e}_s^T\{\tilde{\omega}_s H^s - B\tilde{\omega}_r I_p[B^T\omega_s - \omega_r] + BI_p[\dot{B}^T\omega_s + B^T\dot{\omega}_s - \dot{\omega}_r] + I_s\dot{\omega}_s\} \ ,\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\dot{\omega}_s &= [I_s + BI_pB^T]^{-1}\{-B[\widetilde{(B^T\omega_s)} - \tilde{\omega}_r]I_p[B^T\omega_s - \omega_r] - BI_p[\dot{B}^T\omega_s - \dot{\omega}_r] - \tilde{\omega}_s I_s \omega_s + T_s + BT_p\} \\ &= [I_s + BI_pB^T]^{-1}\{-\omega_s H^s + B\tilde{\omega}_r I_p[B^T\omega_s - \omega_r] - BI_p[\dot{B}^T\omega_s - \dot{\omega}_r] + T_s + BT_p\} \ .\end{aligned} \tag{2.13}$$

Taking the rotor and platform individually as free bodies

$$\dot{\mathbf{H}}_s = \mathbf{e}_s^T[\tilde{\omega}_s I_s \omega_s + I_s \dot{\omega}_s] = \mathbf{e}_s^T[\tilde{\omega}_s H_s + I_s \dot{\omega}_s] = \mathbf{e}_s^T \dot{H}_s = \mathbf{L}_s = \mathbf{e}_s^T L_s \tag{2.14}$$

$$\dot{\omega}_s = I_s^{-1}[-\tilde{\omega}_s I_s \omega_s + L_s] = I_s^{-1}[-\tilde{\omega}_s H_s + L_s] \tag{2.15}$$

$$\dot{\mathbf{H}}_p = \mathbf{e}_p^T[\tilde{\omega}_p I_p \omega_p + I_p \dot{\omega}_p] = \mathbf{e}_p^T[\tilde{\omega}_p H_p + I_p \dot{\omega}_p] = \mathbf{e}_p^T \dot{H}_p = \mathbf{L}_p = \mathbf{e}_p^T L_p \tag{2.16}$$

$$\dot{\omega}_p = I_p^{-1}[-\tilde{\omega}_p I_p \omega_p + L_p] = I_p^{-1}[-\tilde{\omega}_p H_p + L_p] \tag{2.17}$$

where L_s and L_p denote torques applied to the rotor and platform in their respective bases.

Using the kinematic identity $\dot{B} = -\tilde{\omega}_r B$, which can be verified by direct substitution, the following acceleration expressions are obtained from Equations 2.9 and 2.13.

$$\dot{\omega}_p = [I_p + B^T I_s B]^{-1}\{-\tilde{\omega}_p H^p - B^T[\tilde{\omega}_r I_s - I_s \tilde{\omega}_r]B\omega_p - B^T[\tilde{\omega}_r I_s \omega_r + I_s \dot{\omega}_r] + B^T T_s + T_p\}\ , \tag{2.18}$$

and

$$\dot{\omega}_s = [I_s + B I_p B^T]^{-1}\{-\tilde{\omega}_s H^s + B[\tilde{\omega}_r I_p - I_p \tilde{\omega}_r]B^T \omega_s - B[\tilde{\omega}_r I_p \omega_r - I_p \dot{\omega}_r] + T_s + B T_p\}\ . \tag{2.19}$$

Now the terms of Equation 2.18 are listed in detail. H^p has already been given as Equation 1.29. The coefficient of ω_p in the second term is

$$B^T[\tilde{\omega}_r I_s - I_s \tilde{\omega}_r]B = \tilde{\omega}_r B^T I_s B - B^T I_s B \tilde{\omega}_r = \omega_r \begin{bmatrix} 2\bar{I}_{12}^s & -- & -- \\ \bar{I}_{11}^s - \bar{I}_{22}^s & -2\bar{I}_{12}^s & -- \\ \bar{I}_{23}^s & -\bar{I}_{13}^s & 0 \end{bmatrix} \tag{2.20}$$

$$= \omega_r \begin{bmatrix} 2I_{12}^s \cos 2\psi + \Delta I_s \sin 2\psi & ------------ & -- \\ 2I_{12}^s \sin 2\psi - \Delta I_s \cos 2\psi & -2I_{12}^s \cos 2\psi - \Delta I_s \sin 2\psi & -- \\ I_{23}^s \cos\psi + I_{13}^s \sin\psi & -I_{13}^s \cos\psi + I_{23}^s \sin\psi & 0 \end{bmatrix},$$

where $\bar{I}_{ij}^s$ denotes rotor inertia elements expressed in $\mathbf{e}_p$, i.e., the elements of (1.26). Missing terms in (2.20) are supplied by symmetry. The third term in Equation 2.18 is

$$B^T[\tilde{\omega}_r I_s \omega_r + I_s \dot{\omega}_r] = \omega_r^2[\bar{I}_{23}^s, -\bar{I}_{13}^s, 0]^T + \dot{\omega}_r[-\bar{I}_{13}^s, -\bar{I}_{23}^s, \bar{I}_{33}^s]^T \tag{2.21}$$

$$= \omega_r^2 \begin{bmatrix} I_{23}^s \cos\psi + I_{13}^s \sin\psi \\ -I_{13}^s \cos\psi + I_{23}^s \sin\psi \\ 0 \end{bmatrix} + \dot{\omega}_r \begin{bmatrix} -I_{13}^s \cos\psi + I_{23}^s \sin\psi \\ -I_{23}^s \cos\psi - I_{13}^s \sin\psi \\ I_{33}^s \end{bmatrix}.$$

Lastly, the three scalar equations from (2.18) and the 3-axis equation from Equation 2.15 are expanded in detail as Equation 2.22. As noted before all of equations (2.20) through (2.22c) transform to rotor coordinates by exchanging I_p for I_s, ω_s for ω_p , and $-\omega_r$ for ω_r. This exchange in Equation 2.22d produces the 3-axis equation of (2.17).

Some comment on application of torques is in order at this point. When we add $\dot{\mathbf{H}}_p + \dot{\mathbf{H}}_s = \dot{\mathbf{H}}$ in Equation 2.1, all internal torques cancel. Therefore, internal torques, such as the despin torque, do not appear in the various forms of Equation 2.1, e.g., Equation 2.18, 2.19, and 2.22a - c. Instead, internal torques appear as driving torques to the free body equations (2.14), (2.16), or (2.22d). Note that transverse axis internal torques are meaningless, except to determine internal structural loads, as the rotor and platform are constrained to be relatively fixed about these axes. Conversely, external torques do appear in both the appropriate free body equation and the sum, $\dot{\mathbf{H}}$, equation. For example, the Adams[1] dedamper model applies external rotor spin down torque which must appear in (2.22c and d). However, note that a platform external spin torque will appear in Equation 2.22c only.

[1] IDC 4113.10/346, "Dedamper Simulation Model," G. J. Adams, December 4, 1973.

$$[I_{11} + \Delta I_s \sin^2\psi + I^s_{12}\sin 2\psi]\dot{\omega}_{p1} =$$

$$[I^p_{12} + I^s_{12}\cos 2\psi + \frac{\Delta I_s}{2}\sin 2\psi]\dot{\omega}_{p2} + [I^p_{13} + I^s_{13}\cos\psi - I^s_{23}\sin\psi]\dot{\omega}_{p3}$$

$$- I^s_{33}[\omega_{p3} + \omega_r]\omega_{p2} + [I_{22} - I^p_{33} - \Delta I_s \sin^2\psi - I^s_{12}\sin 2\psi]\omega_{p2}\omega_{p3}$$

$$+ [I^p_{13} + I^s_{13}\cos\psi - I^s_{23}\sin\psi]\omega_{p1}\omega_{p2} + [I^p_{23} + I^s_{23}\cos\psi + I^s_{13}\sin\psi]\omega^2_{p2} \tag{2.22a}$$

$$- [I^p_{12} + I^s_{12}\cos 2\psi + \frac{\Delta I_s}{2}\sin 2\psi]\omega_{p1}\omega_{p3} - [I^s_{23}\cos\psi + I^s_{13}\sin\psi][\omega_{p3} + \omega_r]^2 - I^p_{23}\omega^2_{p3}$$

$$- [2I^s_{12}\cos 2\psi + \Delta I_s \sin 2\psi]\omega_r\omega_{p1} - [2I^s_{12}\sin 2\psi - \Delta I_s \cos 2\psi]\omega_r\omega_{p2} + [I^s_{13}\cos\psi - I^s_{23}\sin\psi]\dot{\omega}_r$$

= =

$$[I_{22} - \Delta I_s \sin^2\psi - I^s_{12}\sin 2\psi]\dot{\omega}_{p2} =$$

$$[I^p_{12} + I^s_{12}\cos 2\psi + \frac{\Delta I_s}{2}\sin 2\psi]\dot{\omega}_{p1} + [I^p_{23} + I^s_{23}\cos\psi + I^s_{13}\sin\psi]\dot{\omega}_{p3}$$

$$+ I^s_{33}[\omega_{p3} + \omega_r]\omega_{p1} - [I_{11} - I^p_{33} + \Delta I_s \sin^2\psi + I^s_{12}\sin 2\psi]\omega_{p1}\omega_{p3}$$

$$- [I^p_{23} + I^s_{23}\cos\psi + I^s_{13}\sin\psi]\omega_{p1}\omega_{p2} - [I^p_{13} + I^s_{13}\cos\psi - I^s_{23}\sin\psi]\omega^2_{p1} \tag{2.22b}$$

$$+ [I^p_{12} + I^s_{12}\cos 2\psi + \frac{\Delta I_s}{2}\sin 2\psi]\omega_{p2}\omega_{p3} + [I^s_{13}\cos\psi - I^s_{23}\sin\psi][\omega_{p3} + \omega_r]^2 + I^p_{13}\omega^2_{p3}$$

$$- [2I^s_{12}\sin 2\psi - \Delta I_s \cos 2\psi]\omega_r\omega_{p1} + [2I^s_{12}\cos 2\psi + \Delta I_s \sin 2\psi]\omega_r\omega_{p2} + [I^s_{13}\sin\psi + I^s_{23}\cos\psi]\dot{\omega}_r$$

= =

$$I_{33}\dot{\omega}_{p3} = [I^p_{13} + I^s_{13}\cos\psi - I^s_{23}\sin\psi]\dot{\omega}_{p1}$$

$$+ [I^p_{23} + I^s_{23}\cos\psi + I^s_{13}\sin\psi]\dot{\omega}_{p2}$$

$$[I_{11} - I_{22} + 2\Delta I_s \sin^2\psi + 2I^s_{12}\sin 2\psi]\omega_{p1}\omega_{p2}$$

$$+ I^p_{23}\omega_{p1}\omega_{p3} - I^p_{13}\omega_{p2}\omega_{p3} - I^s_{33}\dot{\omega}_r \tag{2.22c}$$

$$+ [I^p_{12} + I^s_{12}\cos 2\psi + \frac{\Delta I_s}{2}\sin 2\psi][\omega^2_{p1} - \omega^2_{p2}]$$

$$+ [I^s_{23}\cos\psi + I^s_{13}\sin\psi]\omega_{p1}\omega_{p3} - [I^s_{13}\cos\psi - I^s_{23}\sin\psi]\omega_{p2}\omega_{p3}$$

= =

$$I^s_{33}[\dot{\omega}_{p3} + \dot{\omega}_r] = [I^s_{13}\cos\psi - I^s_{23}\sin\psi]\dot{\omega}_{p1} + [I^s_{13}\sin\psi + I^s_{23}\cos\psi]\dot{\omega}_{p2}$$

$$- [\Delta I_s \cos 2\psi - 2I^s_{12}\sin 2\psi]\omega_{p1}\omega_{p2} + [I^s_{13}\sin\psi + I^s_{23}\cos\psi]\omega_{p1}\omega_{p3} \tag{2.22d}$$

$$- [I^s_{13}\cos\psi - I^s_{23}\sin\psi]\omega_{p2}\omega_{p3} - [I^s_{12}\cos 2\psi + \frac{\Delta I_s}{2}\sin 2\psi][\omega^2_{p2} - \omega^2_{p1}]$$

2.1 Simplifications of Dual-Spin Torque Equations

2.1.1 Dynamically Balanced Rotor

Several steps of simplification may be carried out on (2.22) to obtain a much more tractable model for analysis. First we assume rotor dynamic balance, which removes the dominant sinusoidal driving torques of (2.22), then rotor symmetry is imposed which renders the system linear and time-invariant, and decouples the rotor equation (2.22d) from the remaining three. Lastly, requiring the platform to be dynamically balanced as well decouples the spin axis dynamics (2.22c) form the transverse axes.

Rotor dynamic imbalance produces first-order sinusoidal dynamic torques on a the dual-spin vehicle proportional to the product of imbalance and the relative rate squared, ω_r^2. These torques can be identified as the seventh major term listed in Eqs. 2.22a and b. Assuming dynamic balance, let $I_{13}^s = I_{23}^s = 0$, and set $I_{12}^s = 0$ by choice of vector basis subsequent to the dynamic balance condition. Then Eq. 2.21 shows that

$$B^T[\tilde{\omega}_r I_s \omega_r + I_s \dot{\omega}_r]^T = [0,\ 0,\ I_{33}^s \dot{\omega}_r]^T\ . \tag{2.23}$$

Using $\dot{\omega}_r = \dot{\omega}_{s3} - \dot{\omega}_{p3}$ and rewriting Eq. 2.18,

$$[I_p + B^T I_s B]\dot{\omega}_p = -\tilde{\omega}_p H^p - B^T[\tilde{\omega}_r I_s - I_s\tilde{\omega}_r]B\omega_p + B^T T_s + T_p - [0,\ 0,\ I_{33}^s(\dot{\omega}_{s3} - \dot{\omega}_{p3})]^T\ . \tag{2.24}$$

Now let

$$I = [I_p + B^T I_s B] - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{33}^s \end{bmatrix}. \tag{2.25}$$

Then

$$\dot{\omega}_p = I^{-1}\{-\tilde{\omega}_p H^p - B^T[\tilde{\omega}_r I_s - I_s\tilde{\omega}_r]B\omega_p - [0,\ 0,\ I_{33}^s\dot{\omega}_{s3}]^T + B^T T_s + T_p\}\ . \tag{2.26}$$

For rotor spin axis dynamics Equation 2.15 provides

$$\dot{\omega}_{s3} = \omega_{s1}\omega_{s2}[I_{11}^s - I_{22}^s]/I_{33}^s + L_{s3}/I_{33}^s\ . \tag{2.27}$$

2.1.2 Dynamically Balanced and Symmetric Rotor

If in addition to $I_{13}^s = I_{23}^s = 0$ (balance), we have $I_{22}^s - I_{11}^s = \Delta I_s = 0 = I_{12}^s$ (symmetry), then Equations 2.26 and 2.27 reduce to

$$\dot{\omega}_p = I^{-1}\{-\tilde{\omega}_p H^p - [0,\ 0,\ I_{33}^s\dot{\omega}_{s3}]^T\}\ , \tag{2.28}$$

and

$$\dot{\omega}_{s3} = L_{s3}/I_{33}s \tag{2.29}$$

where I is now constant ($B^T I_s B = I_s$) and given as

$$I = \begin{bmatrix} I_{11}^p + I_{11}^s & -I_{12}^p & -I_{13}^p \\ -I_{12}^p & I_{22}^p + I_{22}^s & -I_{23}^p \\ -I_{13}^p & -I_{23}^p & I_{33}^p \end{bmatrix} = \begin{bmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{12} & I_{22} & -I_{23} \\ -I_{13} & -I_{23} & I_{33}^p \end{bmatrix} \tag{2.30}$$

and H^p is greatly simplified to

$$H^p = I\omega_p + [0,\ 0,\ I_{33}^s\omega_{s3}]\ . \tag{2.31}$$

Now let T_1, T_2 be external transverse axis torques, T_3 be the internal spin torque applied to the platform, and T_e^s, T_e^p be respectively external spin torques applied to the rotor and platform. Then expanding Equation 2.31 and 2.28,

$$I_{11}\dot{\omega}_{p1} = I_{12}\dot{\omega}_{p2} + I_{13}\dot{\omega}_{p3} + I_{13}\omega_{p1}\omega_{p2} - I_{12}\omega_{p1}\omega_{p3} + [I_{22} - I_{33}^p]\omega_{p2}\omega_{p3} + I_{23}[\omega_{p2}^2 - \omega_{p3}^2] - I_{33}^s\omega_{s3}\omega_{p2} + T_1 \tag{2.32a}$$

$$I_{22}\dot{\omega}_{p2} = I_{12}\dot{\omega}_{p1} + I_{23}\dot{\omega}_{p3} - I_{23}\omega_{p1}\omega_{p2} + I_{12}\omega_{p2}\omega_{p3} - [I_{11} - I_{33}^p]\omega_{p1}\omega_{p3} + I_{13}[\omega_{p3}^2 - \omega_{p1}^2] + I_{33}^s\omega_{s3}\omega_{p1} + T_2 \tag{2.32b}$$

$$I_{33}^p\dot{\omega}_{p3} = I_{13}\dot{\omega}_{p1} + I_{23}\dot{\omega}_{p2} + I_{23}\omega_{p1}\omega_{p3} - I_{13}\omega_{p2}\omega_{p3} + [I_{11} - I_{22}]\omega_{p1}\omega_{p2} + I_{12}[\omega_{p1}^2 - \omega_{p2}^2] - I_{33}^s\dot{\omega}_{s3} + T_e^s + T_e^p \tag{2.32c}$$

$$I^s_{33}\dot{\omega}_{s3} = -T_3 + T^s_e \ . \tag{2.32d}$$

We note in passing that the rotor dynamic imbalance, or wobble, torque is often reintroduced in (2.32) or (2.34) below by including in torques T_1, T_2 the terms (obtained from the seventh term in 2.22a and b, or the first term of 2.21)

$$[W_1, W_2]^T = -\omega_r^2({I^s_{13}}^2 + {I^s_{23}}^2)^{1/2}[\cos(\omega_r t - \phi), \sin(\omega_r t - \phi)]^T \tag{2.33a}$$

with

$$\phi = \mathrm{Tan}^{-1}\{I^s_{13}/I^s_{23}\} \ . \tag{2.33b}$$

2.1.3 Linearization

To simplify the notation in further reduction of (2.32), we write the platform inertial rate vector as $[\omega_1, \omega_2, \omega_3 + \omega_p]^T$, dropping the sub-p, letting the scalar constant ω_p denote a nominal platform 3-axis rate, and ω_3 represent deviations from this rate. Also, let $\omega_{s3} = \omega_s + \Delta\omega_s$, where ω_s is the constant nominal value. In Appendix A the dual-spin equations are linearized allowing non-zero transverse rates. Here we give the simpler case of linearization about $[0, 0, \omega_p]$ and ω_s which yields

$$I_{11}\dot{\omega}_1 - I_{12}\dot{\omega}_2 - I_{13}\dot{\omega}_3 + I_{12}\omega_p\omega_1 + I_{11}\lambda_1\omega_2 + 2I_{23}\omega_p\omega_3 = -I_{23}\omega_p^2 + T_1 \tag{2.34a}$$

$$I_{22}\dot{\omega}_2 - I_{12}\dot{\omega}_1 - I_{23}\dot{\omega}_3 - I_{12}\omega_p\omega_2 - I_{22}\lambda_2\omega_1 - 2I_{13}\omega_p\omega_3 = I_{13}\omega_p^2 + T_2 \tag{2.34b}$$

$$I^p_{33}\dot{\omega}_3 - I_{13}\dot{\omega}_1 - I_{23}\dot{\omega}_2 + I_{13}\omega_p\omega_2 - I_{23}\omega_p\omega_1 = -I^s_{33}\dot{\omega}_{s3} + T^s_e + T^p_e \tag{2.34c}$$

$$I^s_{33}\dot{\omega}_{s3} = -T_3 + T^s_e \tag{2.34d}$$

where[1]

$$\lambda_1 = [I^s_{33}\omega_s + (I^p_{33} - I_{22})\omega_p]/I_{11} = \sigma_{1e}\omega_s - (I_{22}/I_{11})\omega_p \tag{2.35a}$$

$$\lambda_2 = [I^s_{33}\omega_s + (I^p_{33} - I_{11})\omega_p]/I_{22} = \sigma_{2e}\omega_s - (I_{11}/I_{22})\omega_p \ . \tag{2.35b}$$

Substituting (2.34d) into (2.34c), the resultant linear time-invariant system can be written

$$P(s)^{-1}\omega = T \tag{2.36}$$

where

$$\omega = [\omega_1, \omega_2, \omega_1]^T \tag{2.37}$$

$$T = [T_1 - I_{23}\omega_p^2, T_2 + I_{13}\omega_p^2, T_3 + T^p_e]^T \tag{2.38}$$

and

$$P(s)^{-1} = \begin{bmatrix} [I_{11}s + I_{12}\omega_p] & -[I_{12}s - I_{11}\lambda_1] & -[I_{13}s - 2I_{23}\omega_p] \\ -[I_{12}s + I_{22}\lambda_2] & [I_{22}s - I_{12}\omega_p] & -[I_{23}s + 2I_{13}\omega_p] \\ -[I_{13}s + I_{23}\omega_p] & -[I_{23}s - I_{13}\omega_p] & I^p_{33}s \end{bmatrix}. \tag{2.39}$$

P(s) is the matrix of linearized spacecraft (plant) dynamics. The elements of P are expanded as Eq. 2.40, and a system diagram of the complete linearized rigid body dynamics is shown on Figure 2.1. Equation 2.34d has also been diagramed on Figure 2.1. T_3 is the internal 3-axis (despin) torque and all other torques shown are external.

[1] Note that for $I_{11} = I_{22}$, $\lambda_1 = \lambda_2 = \sigma_e\omega_s - \omega_p = \sigma\omega_{so} - \omega_p = \lambda_o - \omega_p = H/I_T - \omega_p$ assuming momentum conservation in the spin axis (3-axis) during platform spinup.

$$P_{11} = \{[I_{22}I_{33}^{p} - I_{23}^2]s^2 - [I_{12}I_{33}^{p} + I_{13}I_{23}]\omega_p s + 2I_{13}^2\omega_p^2\}/\Delta \tag{2.40a}$$

$$P_{12} = \{[I_{12}I_{33}^{p} + I_{13}I_{23}]s^2 - [I_{33}^{p}I_{11}\lambda_1 + \omega_p(I_{13}^2 + 2I_{23}^2)]s + 2I_{13}I_{23}\omega_p^2\}/\Delta \tag{2.40b}$$

$$P_{13} = \{[I_{13}I_{22} + I_{12}I_{23}]s^2 - [I_{23}I_{11}\lambda_1 + \omega_p(2I_{23}I_{22} - I_{12}I_{13})]s - 2\omega_p[I_{13}I_{11}\lambda_1 - I_{12}I_{23}\omega_p]\}/\Delta \tag{2.40c}$$

$$P_{21} = \{[I_{12}I_{33}^{p} + I_{13}I_{23}]s^2 + [I_{33}^{p}I_{22}\lambda_2 + \omega_p(2I_{13}^2 + I_{23}^2)]s + 2I_{13}I_{23}\omega_p^2\}/\Delta \tag{2.40d}$$

$$P_{22} = \{[I_{11}I_{33}^{p} - I_{13}^2]s^2 + [I_{12}I_{33}^{p} + I_{13}I_{23}]\omega_p s + 2I_{23}^2\omega_p^2\}/\Delta \tag{2.40e}$$

$$P_{23} = \{[I_{23}I_{11} + I_{12}I_{13}]s^2 + [I_{13}I_{22}\lambda_2 + \omega_p(2I_{13}I_{11} - I_{12}I_{23})]s - 2\omega_p[I_{23}I_{22}\lambda_2 - I_{12}I_{13}\omega_p]\}/\Delta \tag{2.40f}$$

$$P_{31} = \{[I_{13}I_{22} + I_{12}I_{23}]s^2 + [I_{23}I_{22}\lambda_2 + \omega_p(I_{23}I_{22} - 2I_{12}I_{13})]s - \omega_p[I_{13}I_{22}\lambda_2 + I_{12}I_{23}\omega_p]\}/\Delta \tag{2.40g}$$

$$P_{32} = \{[I_{23}I_{11} + I_{12}I_{13}]s^2 - [I_{13}I_{11}\lambda_1 + \omega_p(I_{13}I_{11} - 2I_{12}I_{23})]s - \omega_p[I_{23}I_{11}\lambda_1 + I_{12}I_{13}\omega_p]\}/\Delta \tag{2.40h}$$

$$P_{33} = \{[I_{11}I_{22} - I_{12}^2]s^2 + I_{11}I_{22}\lambda_1\lambda_2 - I_{12}^2\omega_p^2\}/\Delta \tag{2.40i}$$

$$\begin{aligned}\Delta(s) = {} & [I_{11}I_{22}I_{33}^{p} - I_{11}I_{23}^2 - I_{22}I_{13}^2 - I_{33}^{p}I_{12}^2 - 2I_{12}I_{13}I_{23}]s^3 \\ & + \{I_{11}I_{22}I_{33}^{p}\lambda_1\lambda_2 + \omega_p[I_{11}\lambda_1(2I_{13}^2 + I_{23}^2) + I_{22}\lambda_2(2I_{23}^2 + I_{13}^2)] \\ & \qquad + \omega_p^2[2I_{13}^2I_{11} + 2I_{23}^2I_{22} - I_{12}^2I_{33}^{p} - 6I_{12}I_{13}I_{23}]\}s \\ & + 2\omega_p^3[I_{13}I_{23}(I_{11} - I_{22}) + I_{12}(I_{13}^2 - I_{23}^2)]\ .\end{aligned} \tag{2.40j}$$

Inverse Laplace transforms of a general plant element for doublet, impulse, step and sinusoidal torque inputs are respectively tabulated below using $\lambda_p^2 = \lambda_1\lambda_2/(1 - r)$ where r is defined in 2.47 below.

Plant Doublet Response

$$sP_{ij} = [As^2 + Bs + C]/[s^2 + \lambda_p^2] = A + [Bs + (C - \lambda_p^2 A)]/[s^2 + \lambda_p^2] \tag{2.41a}$$

$$f(t) = A\delta(t) + B\cos\lambda_p t + [C/\lambda_p - \lambda_p A]\sin\lambda_p t \tag{2.41b}$$

Plant Impulse Response

$$P_{ij} = [As^2 + Bs + C]/[s(s^2 + \lambda_p^2)] = A[s/(s^2 + \lambda_p^2)] + B[1/(s^2 + \lambda_p^2)] + [C/\lambda_p^2][1/s - s/(s^2 + \lambda_p^2)] \tag{2.42a}$$

$$f(t) = [A - C/\lambda_p^2]\cos\lambda_p t + [B/\lambda_p]\sin\lambda_p t + C/\lambda_p^2 \tag{2.42b}$$

Plant Step Response

$$P_{ij}/s = A[1/(s^2 + \lambda_p^2)] + [B/\lambda_p^2][1/s - s/(s^2 + \lambda_p^2)] + [C/\lambda_p^2][1/s^2 - 1/(s^2 + \lambda_p^2)] \tag{2.43a}$$

$$f(t) = [A/\lambda_p - C/\lambda_p^3]\sin\lambda_p t + [B/\lambda_p^2][1 - \cos\lambda_p t] + [C/\lambda_p^2]t \tag{2.43b}$$

Plant Cosine Response

$$F(s) = \left[\frac{As^2 + Bs + C}{s(s^2 + \lambda_p^2)}\right]\left[\frac{s}{s^2 + \omega^2}\right] = \frac{As^2 + Bs + C}{(s^2 + \lambda_p^2)(s^2 + \omega^2)} \tag{2.44a}$$

$$f(t)(\omega^2 - \lambda_p^2) = B[\cos\lambda_p t - \cos\omega t] + [(C - A\lambda_p^2)/\lambda_p]\sin\lambda_p t - [(C - A\omega^2)/\omega]\sin\omega t \tag{2.44b}$$

Plant Sine Response

$$F(s) = \left[\frac{As^2 + Bs + C}{s(s^2 + \lambda_p^2)}\right]\left[\frac{\omega}{s^2 + \omega^2}\right] \tag{2.45a}$$

$$\begin{aligned}f(t)(\omega^2 - \lambda_p^2) = {} & C[(\omega^2 - \lambda_p^2)/\lambda_p^2\omega]u(t) + [(A\lambda_p^2 - C)\omega/\lambda_p^2]\cos\lambda_p t \\ & - [(A\omega^2 - C)/\omega]\cos\omega t + B[(\omega/\lambda_p)\sin\lambda_p t - \sin\omega t]\ .\end{aligned} \tag{2.45b}$$

The part of the system of most frequent interest is the plant elements which respond to despin torque. This portion is diagramed separately on Figure 2.2. In analysis of the despin loop usually $\omega_p = 0$ and $I_{12} = 0$, (or intensionally by choice of basis). Making these simplifications

$$I_{11}\lambda_1 = I_{22}\lambda_2 \,, \tag{2.46}$$

and the plant elements of Figure 2.2 reduce to

$$P_{13} = I_{22}s[I_{13}s - I_{23}\lambda_2]/\Delta \tag{2.47a}$$

$$P_{23} = I_{11}s[I_{23}s + I_{13}\lambda_1]/\Delta \tag{2.47b}$$

$$P_{33} = I_{11}I_{22}[s^2 + \lambda_1\lambda_2]/\Delta \tag{2.47c}$$

with

$$\Delta = I_{11}I_{22}I_{33}^p(1 - r)s[s^2 + \lambda_1\lambda_2/(1 - r)] = I_{11}I_{22}I_{33}^p(1 - r)s[s^2 + \lambda_p^2] \tag{2.47d}$$

$$r = [I_{11}I_{23}^2 + I_{22}I_{13}^2]/I_{11}I_{22}I_{33}^p \,. \tag{2.47e}$$

If in addition $I_{11} = I_{22} = \sqrt{I_{11}I_{22}} = I_T$, then $\lambda_1 = \lambda_2 = \lambda_p = \lambda_o = I_{33}^s\omega_s/I_T$, and corresponding simplifications result is (2.47). Also we often require the plant dynamics for small linear perturbations in rotor to platform relative rate, ω_r, given by

$$\frac{\omega_r}{T_3} = \frac{[\omega_s - \omega_3]}{T_3} = -\,1/I_{33}^s s - P_{33} = \tag{2.47f}$$

$$P_{43} = -\,[I_{11}I_{22}/I_{33}^s]\{[I_{33}^p(1 - r) + I_{33}^s]s^2 + [I_{33}^p(1 - r)\lambda_p^2 + I_{33}^s\lambda_1\lambda_2]\}/\Delta \,.$$

A second case that is significantly simplified results with $\omega_p \neq 0$, and imposing transverse inertia symmetry such that $I_{11} = I_{22} = I_T$, and $I_{12} = 0$. Then $\lambda_1 = \lambda_2 = \lambda = H/I_T - \omega_p = \lambda_o - \omega_p$, and

$$P_{13} = I_T\{I_{13}s^2 - I_{23}[\lambda + 2\omega_p]s - 2I_{13}\lambda\omega_p\}]/\Delta \tag{2.48a}$$

$$P_{23} = I_T\{I_{23}s^2 + I_{13}[\lambda + 2\omega_p]s - 2I_{23}\lambda\omega_p\}]/\Delta \tag{2.48b}$$

$$P_{33} = I_T^2[s^2 + \lambda^2]/\Delta \tag{2.48c}$$

with

$$\Delta = I_T^2 I_{33}^p(1 - r)s\left[s^2 + [\lambda^2 + r(3\lambda\omega_p + 2\omega_p^2)]/(1 - r)\right] = I_T^2 I_{33}^p(1 - r)s[s^2 + \lambda_p^2] \,. \tag{2.48d}$$

The new expressions for λ, λ_p can be substituted in P_{43} to get the relative rate plant with platform motion.

2.1.4 Uncoupled Linearized Spin (3-Axis) Dynamics and Despin Motor Model

Figure 2.3 shows a model of the 3-axis (despin) dynamics with motor and bearing dynamics included. K_v, K_f are motor back emf and viscous friction constants with appropriate units and the motor inductive time constant has been assumed negligibly small. T_c is the control torque command to the motor and some useful relations for command and disturbance inputs are tabulated. This model assumes a statically and dynamically balanced rotor and platform, which is adequate for most preliminary analyses where cross-coupling (nutation, coning, wobble, etc.) is not considered. When cross-coupling is of interest, it can be shown that the effect of the motor and bearing is closely approximated by placing the function $s/(s + \omega_o)$ between T_c and T_3 of Figure 2.2, i.e., the motor-bearing lag displaces the rigid body pole slightly from the origin. Figure 2.4 shows the model of Figure 2.3 embedded in a typical spinning sensor referenced despin control structure. In this structure rotor to platform relative rate (or angle) is measured, say by shaft angle encoder pulses, and fed back in a high bandwidth inner loop through compensation $F_1(s)$. A lower bandwidth outer $L_o(s)$ measures position, perhaps with sun or earth sensor pulses, and closes a position control on platform angle. The local loop around L_o is representative of a phase-lock loop that smooths the sensor pulse train.

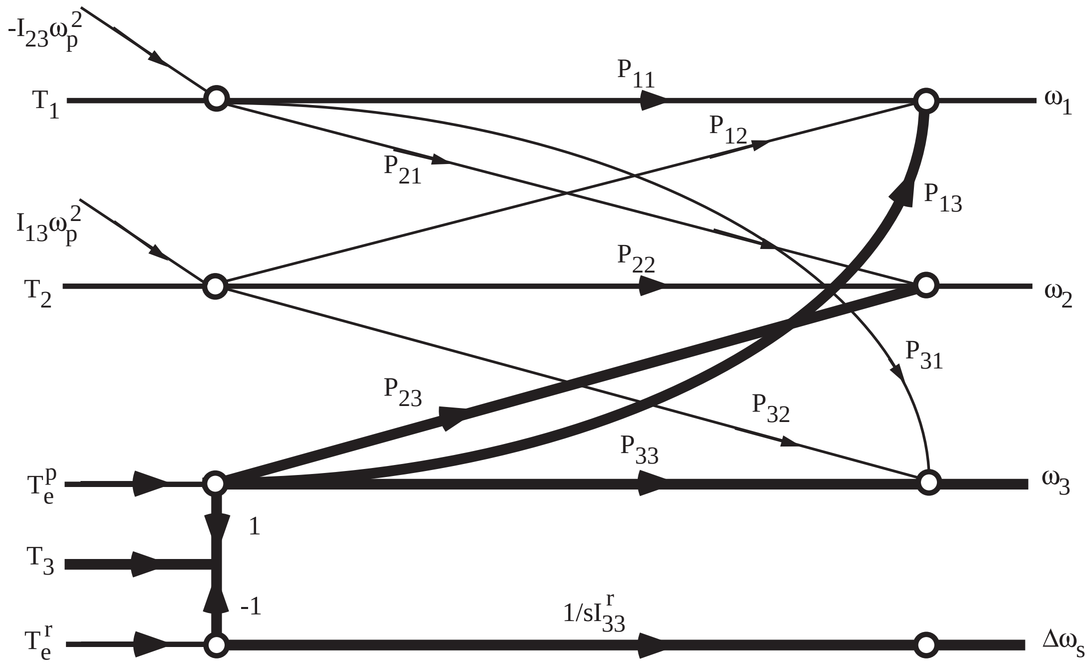


Figure 2.1 Dual-Spin Spacecraft Linearized Rigid Body Dynamics.

$$\Delta = 1 + L(s)$$
$$= 1 + F[P_{33}/s] + G[K_1P_{13} + K_2P_{23}]$$
$$\omega_1(s) = [-FN_1 - GN_2 + D_1]P_{13}/\Delta$$
$$\omega_2(s) = [-FN_1 - GN_2 + D_1]P_{23}/\Delta$$
$$\omega_3(s) = [-FN_1 - GN_2 + D_1][P_{33}/s]/\Delta$$

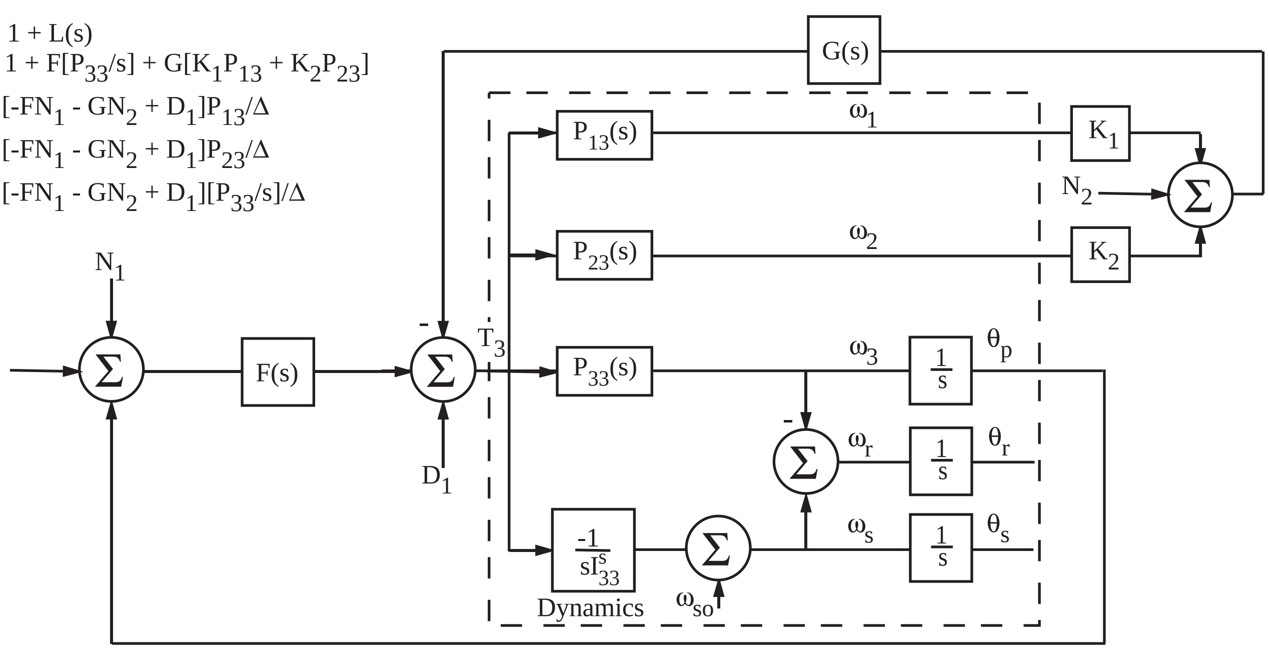
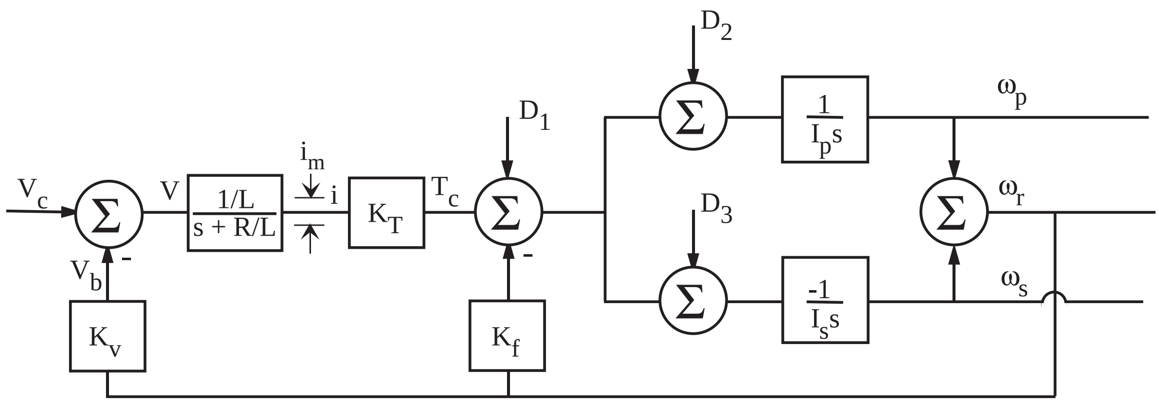


Figure 2.2 Dual-Spin Spacecraft Linearized Rigid Body Spin Axis Dynamics.



$$T_f\,,\ 1/I_s\ = 0 \Rightarrow \omega_p/V_c = \frac{K_T}{(Ls + R)(I_p s + K_f) + K_T K_v}$$

$$\rho = I_p/I_s;\ \ \omega_o = [(K_v + K_f)(1 + \rho)]/I_p;\ \ P_1(s) = 1/[I_p(s + \omega_o)]$$

$$\frac{\omega_p}{T_c} = \frac{\omega_p}{D_1} = P_1;\ \ \frac{\omega_s}{T_c} = \frac{\omega_s}{D_1} = \frac{\omega_r}{D_3} = -\,\rho P_1$$

$$\frac{\omega_r}{T_c} = \frac{\omega_r}{D_1} = -\,(1 + \rho)P_1$$

$$\frac{\omega_p}{D_2} = [s + \omega_o\rho/(1 + \rho)]P_1/s;\ \ \frac{\omega_s}{D_2} = -\,\frac{\omega_p}{D_3} = [\omega_o\rho/(1 + \rho)]P_1/s$$

$$\frac{\omega_s}{D_3} = -\,\rho[s + \omega_o/(1 + \rho)]P_1/s;\ \ \frac{\omega_r}{D_2} = -\,[s + \omega_o\rho^2/(1 + \rho)]P_1/s$$

Figure 2.3 Model of Motor, Bearing, and Spin Axis Dynamics for Dual-Spin Vehicle.

2.1.5 Dedamper Models

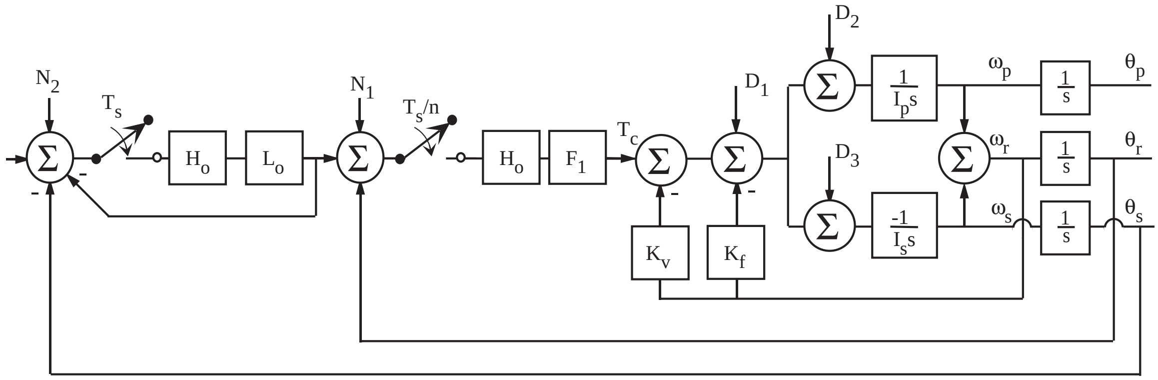
It is frequently useful to model some form of nutation damping or dedamping for analysis or simulation. The simplest such is to feed back a damping torque proportional to transverse rate, say $T_1 = K_1\omega_1 = (2/\tau_d)\omega_1$. This particular model does not conserve angular momentum. The Adams dedamper described next does not conserve momentum instantaneously, but does conserve it over one spin cycle, or on the average. It is a very simple model that is efficient and easy to use, and conserve momentum adequately for most applications. It has a structure that drives energy dissipation to zero in flat spin when nutation angle goes to 90^o. The spherical dedamper, originated bu Kane, is a viscously coupled sphere at the body (rotor in this model) cm that will continue to remove energy in flat spin if excited.

Adams Dedamper

Recorded here for convenient reference is a frequently useful dedamper/damper model first proposed by Jerry Adams,

$$\mathbf{T}_d = \mathbf{e}_p^T[T_1,\, T_2,\, T_e^s]^T = \mathbf{e}_p^T(1/\tau_d)[H_3/H]^2[-H_1(H_3/H),\, -\,H_2(H_3/H),\, H_T(H_T/H)]^T \tag{2.49}$$

$$= \mathbf{e}_p^T(1/\tau_d)\cos^2\theta_n[-H_1 \cos\theta_n,\, -\,H_2 \cos\theta_n,\, H_T \sin\theta_n]^T\ .$$



$$\rho = I_p/I_s;\ \ \omega_o = [(K_v + K_f)(1+\rho)]/I_p;\ \ P(s) = 1/[I_p(s+\omega_o)]$$

$$L_1 = (1+\rho)F_1P = \text{inner loop transmission with respect } \theta_r$$

$$L_2 = -[\rho/(1+\rho)][L_1/(1+L_1)][L_o/(1+L_o)] = \text{System open} - \text{loop transmission with respect } \theta_s$$

$$T_{N_1} = \theta_p/N_1 = [L_1/(1+\rho)]/[(1+L_1)(1+L_2)]$$

$$T_{N_2} = \theta_p/N_2 = -[1/\rho][L_2/(1+L_2)]$$

$$T_{D_1} = \theta_p/D_1 = P/[(1+L_1)(1+L_2)]$$

$$T_{D_2} = \theta_p/D_2 = [P/s]\{L_2(1+L_1)[s+(1+\rho+\rho^2)\omega_o/(1+\rho)] + (1+L_1)(s+\rho\omega_o)$$
$$-[L_1/(1+\rho)][s+\rho^2\omega_o/(1+\rho)]\}/[(1+L_1)(1+L_2)]$$

$$T_{D_3} = \theta_p/D_3 = [\rho/(1+\rho)][P/s][\omega_o + (L_1/L_o)(s+\omega_o)]/[(1+L_1)(1+L_2)]$$

Figure 2.4 Decoupled Spin Axis Dynamics and Sampled Spinning Sensor Control Structure with Phase Lock Loop.

This function is defined such that $\mathbf{T}_d \cdot \mathbf{H} = 0$ so a transverse torque normal to $\mathbf{H}$ and co-aligned with the current H_T increases this component and a spin down component is applied to the rotor. Conservation of momentum by the damping torque is shown by

$$\frac{d|\mathbf{H}|^2}{dt} = \frac{d[\mathbf{H}\cdot\mathbf{H}]}{dt} = 2\dot{\mathbf{H}}\cdot\mathbf{H} = 2\mathbf{T}_d\cdot\mathbf{H} = 0\ . \tag{2.50}$$

Spherical Dedamper

Consider a simple two body system comprising a spacecraft rotor with unconstrained mass properties and a spherical mass located coincident with the center of mass of the first body. The spherical mass is a damper/dedamper fixed in position with the rotor and constrained in angular rate by viscous damping. Denote the damper angular velocity as $\boldsymbol{\omega}_d$, and the relative velocity as $\boldsymbol{\nu} = \boldsymbol{\omega}_d - \boldsymbol{\omega}_s$. $\boldsymbol{\nu} = \boldsymbol{\omega}_d - (\boldsymbol{\omega}_p + \boldsymbol{\omega}_r)$. Then the rotor torque equation may be written

$$\dot{\mathbf{H}}_s = \int \mathbf{r}\times\ddot{\mathbf{r}}dm = \mathbf{J}_s\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times\mathbf{H}_s = \mathbf{J}_s\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times[\mathbf{J}_s\cdot\boldsymbol{\omega}_s] = C_d\boldsymbol{\nu} + \mathbf{T}_s \tag{2.51}$$

where $C_d\boldsymbol{\nu}$ is viscous damper interaction torque. Similarly the damper equation is

$$\dot{\mathbf{H}}_d = \mathbf{J}_d\cdot\dot{\boldsymbol{\omega}}_d + \boldsymbol{\omega}_d\times\mathbf{H}_d = \mathbf{J}_d\cdot\dot{\boldsymbol{\omega}}_d + \boldsymbol{\omega}_d\times[\mathbf{J}_d\cdot\boldsymbol{\omega}_d] = -C_d\boldsymbol{\nu}\ . \tag{2.52}$$

Using the diagonal and symmetry properties of the spherical inertia dyadic $\mathbf{J}_d = \mathbf{e}_s^T J_d \mathbf{e}_s = J_d \mathbf{e}_s^T \mathbf{I} \mathbf{e}_s$ where I denotes the identity matrix, and noting that $\boldsymbol{\omega}_d\times\boldsymbol{\omega}_d = 0$

$$\dot{\mathbf{H}}_s = \int \mathbf{r}\times\ddot{\mathbf{r}}dm = \mathbf{J}_s\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times\mathbf{H}_s = \mathbf{J}_s\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times[\mathbf{J}_s\cdot\boldsymbol{\omega}_s] = C_d\boldsymbol{\nu} + \mathbf{T}_s \tag{2.51}$$

where $C_d\boldsymbol{\nu}$ is viscous damper interaction torque. Similarly the damper equation is

$$\dot{\mathbf{H}}_d = \mathbf{J}_d\cdot\dot{\boldsymbol{\omega}}_d + \boldsymbol{\omega}_d\times\mathbf{H}_d = \mathbf{J}_d\cdot\dot{\boldsymbol{\omega}}_d + \boldsymbol{\omega}_d\times[\mathbf{J}_d\cdot\boldsymbol{\omega}_d] = -C_d\boldsymbol{\nu}\ . \tag{2.52}$$

Using the diagonal and symmetry properties of the spherical inertia dyadic $\mathbf{J}_d = \mathbf{e}_s^T J_d \mathbf{e}_s = J_d \mathbf{e}_s^T \mathbf{I}\mathbf{e}_s$ where I denotes the identity matrix, and noting that $\boldsymbol{\omega}_d\times\boldsymbol{\omega}_d = 0$

$$\dot{\mathbf{H}}_d = \mathbf{J}_d\cdot\dot{\boldsymbol{\omega}}_d = \mathbf{J}_d\cdot(\dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\nu}}) = \mathbf{J}_d\cdot(\dot{\boldsymbol{\omega}}_p + \dot{\boldsymbol{\omega}}_r + \dot{\boldsymbol{\nu}}) = -C_d\boldsymbol{\nu}\ . \tag{2.52}$$

Since $\mathbf{J}_d$ is invariant under orthogonal coordinate transformations we can expand $\boldsymbol{\omega}_d$ in any convenient basis. Extending to three bodies and summing momentum

$$\begin{aligned}\dot{\mathbf{H}} = \dot{\mathbf{H}}_p + \dot{\mathbf{H}}_s + \dot{\mathbf{H}}_d &= \dot{\mathbf{H}}_p + [\mathbf{J}_s + \mathbf{J}_d]\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times[\mathbf{J}_s\cdot\boldsymbol{\omega}_s] \\ &= \dot{\mathbf{H}}_p + [\mathbf{J}_s + \mathbf{J}_d]\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times[(\mathbf{J}_s + \mathbf{J}_d)\cdot\boldsymbol{\omega}_s] = -\mathbf{J}_d\cdot\dot{\boldsymbol{\nu}} + \mathbf{T}_s + \mathbf{T}_p \\ &= \dot{\mathbf{H}}_p + \mathbf{I}_s\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times[\mathbf{I}_s\cdot\boldsymbol{\omega}_s] = -\mathbf{J}_d\cdot\dot{\boldsymbol{\nu}} + \mathbf{T}_s + \mathbf{T}_p\end{aligned} \tag{2.54}$$

where $\mathbf{I}_s$ is the inertia dyadic of the combined rotor and damper. Now assigning scalar variables to the elements of ν and denoting the time derivative with respect to the rotor fixed frame $\mathbf{e}_s$ as with the presuperscript s

$$\boldsymbol{\nu} = \mathbf{e}_s^T[\nu_1, \nu_2, \nu_3]^T\ \ ;\ \frac{{}^s d\boldsymbol{\nu}}{dt} = \mathbf{e}_s^T[\dot{\nu}_1, \dot{\nu}_2, \dot{\nu}_3]^T \tag{2.55a}$$

$$\boldsymbol{\nu} = \mathbf{e}_p^T[u_1, u_2, u_3]^T\ \ ;\ \frac{{}^p d\boldsymbol{\nu}}{dt} = \mathbf{e}_s^T[\dot{u}_1, \dot{u}_2, \dot{u}_3]^T \tag{2.55b}$$

$$\dot{\boldsymbol{\nu}} = \frac{{}^s d\boldsymbol{\nu}}{dt} + \boldsymbol{\omega}_s\times\boldsymbol{\nu} = \mathbf{e}_s^T\begin{bmatrix}\dot{\nu}_1 + \omega_{s2}\nu_3 - \omega_{s3}\nu_2 \\ \dot{\nu}_2 + \omega_{s3}\nu_1 - \omega_{s1}\nu_3 \\ \dot{\nu}_3 + \omega_{s1}\nu_2 - \omega_{s2}\nu_1\end{bmatrix} = \frac{{}^p d\boldsymbol{\nu}}{dt} + \boldsymbol{\omega}_p\times\boldsymbol{\nu} = \mathbf{e}_p^T\begin{bmatrix}\dot{u}_1 + \omega_{p2}u_3 - \omega_{p3}u_2 \\ \dot{u}_2 + \omega_{p3}u_1 - \omega_{p1}u_3 \\ \dot{u}_3 + \omega_{p1}u_2 - \omega_{p2}u_1\end{bmatrix} \tag{2.56}$$

while

$$\dot{\boldsymbol{\omega}}_r = \frac{{}^p d\boldsymbol{\omega}_r}{dt} + \boldsymbol{\omega}_p\times\boldsymbol{\omega}_r = \mathbf{e}_p^T[\omega_{p2}\omega_r, -\omega_{p1}\omega_r, \dot{\omega}_r]^T = \frac{{}^s d\boldsymbol{\omega}_r}{dt} + \boldsymbol{\omega}_s\times\boldsymbol{\omega}_r = \mathbf{e}_s^T[\omega_{s2}\omega_r, -\omega_{s1}\omega_r, \dot{\omega}_r]^T\ . \tag{2.57}$$

The damper equation (2.52) expands respectively in a rotor or platform basis as

$$\mathbf{J}_d\cdot(\dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\nu}}) = \mathbf{e}_s^T J_d\begin{bmatrix}\dot{\omega}_{s1} + \dot{\nu}_1 + \omega_{s2}\nu_3 - \omega_{s3}\nu_2 \\ \dot{\omega}_{s2} + \dot{\nu}_2 + \omega_{s3}\nu_1 - \omega_{s1}\nu_3 \\ \dot{\omega}_{s3} + \dot{\nu}_3 + \omega_{s1}\nu_2 - \omega_{s2}\nu_1\end{bmatrix} = -\mathbf{e}_s^T C_d\begin{bmatrix}\nu_1 \\ \nu_2 \\ \nu_3\end{bmatrix} \tag{2.58a}$$

$$\mathbf{J}_d\cdot(\dot{\boldsymbol{\omega}}_p + \dot{\boldsymbol{\omega}}_r + \dot{\boldsymbol{\nu}}) = \mathbf{e}_p^T J_d\begin{bmatrix}\dot{\omega}_{p1} + \dot{u}_1 + \omega_{p2}(\omega_r + u_3) - \omega_{p3}u_2 \\ \dot{\omega}_{p2} + \dot{u}_2 + \omega_{p3}u_1 - \omega_{p1}(\omega_r + u_3) \\ \dot{\omega}_{p3} + \dot{\omega}_r + \dot{u}_3 + \omega_{p1}u_2 - \omega_{p2}u_1\end{bmatrix} = -\mathbf{e}_p^T C_d\begin{bmatrix}u_1 \\ u_2 \\ u_3\end{bmatrix}. \tag{2.58b}$$

Note that spherical symmetry of the damping coefficient C_d is implicit in the right side of the last form.

Hence to add the spherical damper to a single body spinner, set $\mathbf{H}_p = 0$, introduce the three damper equations of (2.58a) and add the first form of $\mathbf{J}_d\cdot\boldsymbol{\nu}$ from (2.56) in (2.54). For a dual-spin, if one wishes to operate in platform coordinates, use the second form of damper equations, (2.58b), augment (2.54) or (2.22a) through c with the second form of $\mathbf{J}_d\cdot\boldsymbol{\nu}$ from (2.56) and repeat the third element of (2.56) as a reaction torque in the fourth equation 2.22d.

An approximate time constant developed by Murphy/Jennings assuming static balance of both bodies and $\omega_p = u_3 = 0$ is

$$\tau_d = -\frac{I_{33}^s(C_d^2 + \lambda_o^2 J_d^2)}{C_d\omega_s\lambda_s[J_d I_{33}^s/I_T]^2} = \frac{(I_{33}^s/J_d)[1 + (\lambda_o\tau)^2]}{\tau\omega_s\lambda_s\sigma^2}\ , \tag{2.59}$$

where $\lambda_o = H/I_T$, and $\lambda_s = \lambda_o - \omega_s$. In practice one will probably choose J_d and solve for C_d to get the desired time constant τ_d as

$$C_d = \frac{-\tau_d\omega_s\lambda_s[J_d I_{33}^s/I_T]^2}{[2I_{33}^s]}\left[1 \pm \sqrt{1 - \{[2\lambda_o I_T^2]/[\tau_d\omega_s\lambda_s J_d I_{33}^s]\}^2}\,\right] = -(\tau_d/2)\omega_s\lambda_s I_{33}^s[J_d/I_T]^2\left[1 \pm \sqrt{1 - \{[2I_T]/[\tau_d\lambda_s J_d]\}^2}\,\right]. \qquad (2.60)$$

In order that C_d be a real number damper inertia is bounded below as

$$J_d > J_{min} = \frac{2\lambda_o I_T^2}{\tau_d\omega_s|\lambda_s|I_{33}^s} = \frac{2I_T}{\tau_d|\lambda_s|} = \frac{2I_{33}^s}{\tau_d\sigma|\lambda_s|} \qquad (2.61)$$

and choosing $J_d = \alpha J_{min}$ gives

$$C_d = \frac{-\tau_d\omega_s\lambda_s[J_d I_{33}^s/I_T]^2}{[2I_{33}^s]}\left[\frac{\alpha \pm \sqrt{\alpha^2 - 1}}{\alpha}\right] = -2\alpha\left[\alpha \pm \sqrt{\alpha^2 - 1}\right]I_{33}^s[\omega_s/(\tau_d\lambda_s)]\ , \qquad (2.62)$$

and rather arbitrairly for $\alpha = 2$,

$$C_d = -4[2 \pm \sqrt{3}]I_{33}^s[\omega_s/(\tau_d\lambda_s)]\ . \qquad (2.63)$$

3.0 Generalizations for Static Imbalance

Static balance on a dual-spin spacecraft means that the body mass centers lie on the bearing axis. This case is analytically very convenient because the position vectors of the body mass centers with respect to the vehicle cm $\mathbf{r}_s$, $\mathbf{r}_p$ have only components along the common bearing axis and are therefore fixed in both the rotor and platform. The equations developed to this point are restricted to this case. We shall find below that if one body only is statically imbalanced, the torque equations have exactly the same form when written in the unbalanced body, but must employ a modified inertia formulation to account for the imbalance. This generalization is correct for the full nonlinear time-varying model when only one body has imbalance. When both bodies are statically imbalanced, the system is hopelessly time-varying. However, even for this case a small angle linear time-invariant approximation is obtained by applying approximate sinusoidal imbalance torques. For large angle motion simulation is the approach.

3.1 Addition of Platform Static Imbalance

In Equation 1.3 above, ${}^s d\mathbf{r}_s/dt = 0$ has been assumed, i.e., $\mathbf{r}_s$ is fixed in the rotor basis $\mathbf{e}_s$. Removing this assumption by allowing a platform static imbalance, the total vehicle cm will be displaced from the bearing axis and will no longer remain fixed with respect to the rotor. Additional equations are now developed to treat this case.

Repeating the rotor momentum expansion analogous to Eq. 1.3b yields

$$\mathbf{H}_s = \int [\mathbf{r}_s + \boldsymbol{\mu}_s]\times[\dot{\mathbf{r}}_s + \dot{\boldsymbol{\mu}}_s]dm = \mathbf{J}_s \cdot \boldsymbol{\omega}_s + m_s\mathbf{r}_s\times\dot{\mathbf{r}}_s \tag{3.1}$$

$$= \mathbf{J}_s \cdot \boldsymbol{\omega}_s - m_s\mathbf{r}_s\times[\mathbf{r}_s\times\boldsymbol{\omega}_s] + m_s\mathbf{r}_s\times\frac{{}^s d\mathbf{r}_s}{dt} = \mathbf{I}_s\cdot\boldsymbol{\omega}_s + m_s\mathbf{r}_s\times\frac{{}^s d\mathbf{r}_s}{dt}\ ,$$

where $\mathbf{J}_s$, $\mathbf{I}_s$ are the rotor inertia dyadics with respect to the rotor and vehicle cm, respectively. Inertial time derivatives are denoted $\dot{\mathbf{v}}$, while ${}^s d\mathbf{v}/dt$ denotes differentiation with respect to rotor basis $\mathbf{e}_s$. Comparing with Equation 1.5, it is seen that the ${}^s d\mathbf{r}_s/dt$ term has appeared and, less obvious, $\mathbf{I}_s(t)$ is no longer fixed in $\mathbf{e}_s$. Differentiating a second time to get the rotor torque equation,

$$\dot{\mathbf{H}}_s = \int [\mathbf{r}_s + \boldsymbol{\mu}_s]\times[\ddot{\mathbf{r}}_s + \ddot{\boldsymbol{\mu}}_s]dm = \mathbf{J}_s\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times[\mathbf{J}_s\cdot\boldsymbol{\omega}_s] + m_s\mathbf{r}_s\times\ddot{\mathbf{r}}_s$$

$$= \{\mathbf{J}_s\cdot\dot{\boldsymbol{\omega}}_s - m_s\mathbf{r}_s\times[\mathbf{r}_s\times\dot{\boldsymbol{\omega}}_s]\} + \{\boldsymbol{\omega}_s\times[\mathbf{J}_s\cdot\boldsymbol{\omega}_s] - m_s\mathbf{r}_s\times[\boldsymbol{\omega}_s\times(\mathbf{r}_s\times\boldsymbol{\omega}_s)]\}$$

$$+ \{2m_s\mathbf{r}_s\times[\boldsymbol{\omega}_s\times\frac{{}^s d\mathbf{r}_s}{dt}]\} + \{m_s\mathbf{r}_s\times\frac{{}^s d^2\mathbf{r}_s}{dt^2}\} \tag{3.2}$$

$$= \{\mathbf{I}_s\cdot\dot{\boldsymbol{\omega}}_s\} + \{\boldsymbol{\omega}_s\times[\mathbf{I}_s\cdot\boldsymbol{\omega}_s]\} + \{\frac{{}^s d\mathbf{I}_s}{dt}\cdot\boldsymbol{\omega}_s + m_s[\mathbf{r}_s\times(\boldsymbol{\omega}_s\times\frac{{}^s d\mathbf{r}_s}{dt}) - \frac{{}^s d\mathbf{r}_s}{dt}\times(\boldsymbol{\omega}_s\times\mathbf{r}_s)]\}$$

$$+ \{m_s\mathbf{r}_s\times\frac{{}^s d^2\mathbf{r}_s}{dt}\}\ .$$

The brackets in (3.2) indicate sequentially equal terms. This equation is the general expression of the derivative of rotor momentum when the vehicle cm is not fixed in $\mathbf{e}_s$, i.e., in the rotor.

Statically Balanced Rotor

Now consider the case where the rotor is statically balanced. When this constraint holds, $\mathbf{r}_s$ is fixed in $\mathbf{e}_p$, and using the definition for the vehicle cm it can be written

$$\mathbf{r}_s = -(m_p/m_s)\mathbf{r}_p\ . \tag{3.3}$$

The inertial derivatives may then be expressed

$$\dot{\mathbf{r}}_s = \boldsymbol{\omega}_p\times\mathbf{r}_s \tag{3.4a}$$

$$\ddot{\mathbf{r}}_s = \dot{\boldsymbol{\omega}}_p\times\mathbf{r}_s + \boldsymbol{\omega}_p\times[\boldsymbol{\omega}_p\times\mathbf{r}_s]\ . \tag{3.4b}$$

Substituting (3.4b) into the second expression of (3.2),

$$\dot{\mathbf{H}}_s = \mathbf{J}_s\cdot\dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s\times[\mathbf{J}_s\cdot\boldsymbol{\omega}_s] - m_s\mathbf{r}_s\times[\mathbf{r}_s\times\dot{\boldsymbol{\omega}}_p] - m_s\mathbf{r}_s\times[\boldsymbol{\omega}_p\times(\mathbf{r}_s\times\boldsymbol{\omega}_p)]\ . \tag{3.5}$$

Noting that $\mathbf{r}_p$ is fixed in $\mathbf{e}_p$, the platform momentum is unchanged from the form previously obtained. Thus, adding $\dot{\mathbf{H}}_p$ to (3.5),

$$\dot{\mathbf{H}} = \mathbf{J}_s \cdot \dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s \times [\mathbf{J}_s \cdot \boldsymbol{\omega}_s] + \mathbf{I}_p \cdot \dot{\boldsymbol{\omega}}_p - m_s \mathbf{r}_s \times [\mathbf{r}_s \times \dot{\boldsymbol{\omega}}_p] + \boldsymbol{\omega}_p \times [\mathbf{I}_p \cdot \boldsymbol{\omega}_p] - m_s \mathbf{r}_s \times [\boldsymbol{\omega}_p \times (\mathbf{r}_s \times \boldsymbol{\omega}_p)]$$

$$= \mathbf{J}_s \cdot \dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s \times [\mathbf{J}_s \cdot \boldsymbol{\omega}_s] + \hat{\mathbf{I}}_p \cdot \dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p \times [\hat{\mathbf{I}}_p \cdot \boldsymbol{\omega}_p] \ , \tag{3.6}$$

where $\mathbf{J}_s$, $\mathbf{J}_p$ are the rotor and platform inertia dyadics with respect to body mass centers, and the platform inertia dyadic with respect to the vehicle cm, $\mathbf{I}_p = \mathbf{e}_p^T [J_p - m_p \tilde{r}_p \tilde{r}_p] \mathbf{e}_p$, is replaced by

$$\hat{\mathbf{I}}_p = \mathbf{e}_p^T [J_p - m_p \tilde{r}_p \tilde{r}_p - m_s \tilde{r}_s \tilde{r}_s] \mathbf{e}_p = \mathbf{e}_p^T [J_p - m_p (1 + m_p/m_s) \tilde{r}_p \tilde{r}_p] \mathbf{e}_p = \mathbf{e}_p^T [I_p - m_s \tilde{r}_s \tilde{r}_s] \mathbf{e}_p \ . \tag{3.7}$$

Note now that (3.7) is identical in form to Eq. 2.3 from which the scalar expansion of (2.22) is eventually obtained. This is a very pleasing result, as it means that all the previous expanded equations for derivative of total system momentum can be generalized to the statically balanced rotor and arbitrarily unbalanced platform. This is done by using in prior equations the rotor inertia dyadic about the rotor cm, J_s and the *generalized* inertia of (3.7) for platform inertia. In particular, Eq. 2.22a-c may be so generalized. Note that the rotor need *not* be dynamically balanced.

To obtain the fourth equation necessary to completely describe the four-degree-of-freedom vehicle, we can equate $\dot{\mathbf{H}}_s$ to the moments applied to the rotor as a free body. Assuming no external forces on the rotor or platform, the force applied to the rotor by the platform is

$$\mathbf{F}_b = -m_p \ddot{\mathbf{r}}_p \ . \tag{3.8}$$

Denoting the position vector to the point of force application (the despin bearing center of symmetry) with respect to the vehicle cm by $\mathbf{r}_b$, the moment on the rotor is

$$\mathbf{M}_b = -m_p \mathbf{r}_b \times \ddot{\mathbf{r}}_p \ . \tag{3.9}$$

Equating this to the second form of (3.1) yields

$$\mathbf{J}_s \cdot \dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s \times [\mathbf{J}_s \cdot \boldsymbol{\omega}_s] - m_p [\mathbf{r}_s - \mathbf{r}_b] \times \ddot{\mathbf{r}}_p = 0 \ . \tag{3.10}$$

Now we are interested only in the 3-axis equation from (3.10). If the rotor is statically balanced $\mathbf{r}_s - \mathbf{r}_b$ has only a 3-axis component. Therefore, the last term in (3.10) can make no contribution to the 3-axis equation and indeed the scalar expansion of (3.10) is given by (2.22d) with I_s replaced by J_s.

To summarize, for a statically balanced rotor and arbitrary platform, the four vehicle torque equations are obtained by replacing $I_s = J_s - m_s \tilde{r}_s \tilde{r}_s$ with J_s in (2.22) and replacing

$$I_p = J_p - m_p \tilde{r}_p \tilde{r}_p \tag{3.11}$$

with

$$\hat{I}_p = I_p - (m_p^2/m_s) \tilde{r}_p \tilde{r}_p = J_p - m_p (1 + m_p/m_s) \tilde{r}_p \tilde{r}_p = J_p - m_p \tilde{r}_p \tilde{r}_p - m_s \tilde{r}_s \tilde{r}_s \tag{3.12}$$

in Equations 2.22a through c.

All to the results up to and including (2.40) hold with stated assumptions and substitution of the appropriate inertia parameters. Equations (2.46) and (2.47) are not in general valid because $\hat{I}_{12} \neq 0$ with a statically imbalanced platform.

It is convenient to note another interpretation of $\hat{I}_p$. The total spacecraft inertia at some relative phase may be written

$$I = J_p - m_p \tilde{r}_p \tilde{r}_p + J_s - m_s \tilde{r}_s \tilde{r}_s \ . \tag{3.13}$$

Using the definition of vehicle cm

$$m_s \mathbf{r}_s = -m_p \mathbf{r}_p \tag{3.14}$$

and substituting r_s in I yields

$$\hat{I}_p = I - J_s \,, \tag{3.15}$$

hence the augmented platform inertia matrix is the total vehicle inertia about the vehicle cm minus the rotor inertia about the rotor cm. Also the platform inertia with respect to the rotor cm is $J_p - m_p(1 + m_p/m_s)^2\, \tilde{r}_p \tilde{r}_p \neq \hat{I}_p$.

The moments that are equated to (3.6) and (3.10) in the presence of platform static imbalance are also altered. Let $\mathbf{x}_i$ denote the positions of application with respect to the platform cm of forces $\mathbf{F}_i^p$ on the platform, and similarly $\mathbf{y}_i$, $\mathbf{F}_i^s$ on the rotor. Also, let $\mathbf{T}_p$, $\mathbf{T}_s$ represent pure torques applied to the platform and rotor. Then, without supplying details, the torque equations corresponding to (3,6) and (3.10) respectively are

$$\hat{\mathbf{I}}_p \cdot \dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p \times [\hat{\mathbf{I}}_p \cdot \boldsymbol{\omega}_p] + \mathbf{J}_s \cdot \dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s \times [\mathbf{J}_s \cdot \boldsymbol{\omega}_s] = \mathbf{T}_p + \mathbf{T}_s + \Sigma[\mathbf{r}_p + \mathbf{x}_i] \times \mathbf{F}_i^p + \Sigma[\mathbf{r}_s + \mathbf{y}_i] \times \mathbf{F}_i^s \tag{3.16}$$

$$\mathbf{J}_s \cdot \dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s \times [\mathbf{J}_s \cdot \boldsymbol{\omega}_s] = \mathbf{T}_s + \Sigma \mathbf{y}_i \times \mathbf{F}_i^s + [\mathbf{r}_s - \mathbf{r}_b] \times [m_p \ddot{\mathbf{r}}_p - (m_s/m)\Sigma \mathbf{F}_i^p + (m_p/m)\Sigma \mathbf{F}_i^s] \,. \tag{3.17}$$

Again, the last term in (3.17) makes no contribution to the 3-axis scalar equation. However, (3.16) has the term $\mathbf{r}_s \times \Sigma \mathbf{F}_i^s$. Since this term appears in the total momentum derivative (3.16), but not in the rotor component (3.17), it has the form of an external platform torque (D_2 in Figure 2.3, page 2.8) even though it arises from a force on the rotor.

It is informative to inspect the platform free body torque equation which (3.17) parallels. This equation is

$$\mathbf{J}_p \cdot \dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p \times [\mathbf{J}_p \cdot \boldsymbol{\omega}_p] = \mathbf{T}_p + \Sigma \mathbf{x}_i \times \mathbf{F}_i^p + [\mathbf{r}_p - \mathbf{r}_b] \times [m_s \ddot{\mathbf{r}}_s - (m_p/m)\Sigma \mathbf{F}_i^s + (m_s/m)\Sigma \mathbf{F}_i^p] \,. \tag{3.18}$$

Here the 1 and 2-axis components of $\mathbf{r}_p - \mathbf{r}_b$ do not vanish when the rotor is statically balanced and the platform is statically imbalanced. If $\Sigma \mathbf{F}_i^p = 0$, then $\Sigma F_i^s = m\ddot{\mathbf{r}}_o$, and the last term in (3.18) becomes $m_p[\mathbf{r}_b - \mathbf{r}_p] \times [\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_p]$ which is, as one should anticipate, the moment due to the platform acceleration force applied at the bearing.

Finally, the momentum change due to a force applied to the rotor of a dual-spin vehicle can be shown to be

$$\Delta \mathbf{H} = [\mathbf{r}_s + \mathbf{y}] \times \int \mathbf{F}^s dt \,. \tag{3.19}$$

For example, if a spin thruster is fired impulsively on a dual-spin vehicle with statically unbalanced platform initially and finally despun, the rotor spin torque depends upon platform position at the time of firing. Sometimes an unbalanced platform will be positioned to get the desired combination of momentum change and radial velocity change from a radial thrusting maneuver.

3.2 Combined Rotor and Platform Static Imbalance

Let $\mathbf{r}_1$, $\mathbf{r}_2$ denote the position of rotor and platform mass centers with respect to the vehicle cm when the rotor is statically balanced. This is consistent with the notation of previous derivations. If we now introduce a rotor static imbalance by displacing the rotor cm by

$$\mathbf{r}_e = \mathbf{e}_s^T [x_s, y_s, 0]^T \,, \tag{3.20}$$

the respective position vectors from the vehicle cm to body centers become

$$\mathbf{r}_s = (1 - m_s/m)\mathbf{r}_e + \mathbf{r}_1 \tag{3.21}$$

$$\mathbf{r}_p = -(1 - m_p/m)\mathbf{r}_e + \mathbf{r}_2 = -(m_s/m_p)\mathbf{r}_s \,. \tag{3.22}$$

The geometry is illustrated on Figure 3.1. $\mathbf{r}_1$ and $\mathbf{r}_2$ are fixed in $\mathbf{e}_p$, and expressed as

$$\mathbf{r}_1 = \mathbf{e}_p^T [-(1 - m_s/m)x_p, -(1 - m_s/m)y_p, z_s]^T \tag{3.23}$$

$$\mathbf{r}_2 = \mathbf{e}_p^T [(1 - m_p/m)x_p, (1 - m_p/m)y_p, z_p]^T = -(m_s/m_p)\mathbf{r}_1 \,. \tag{3.24}$$

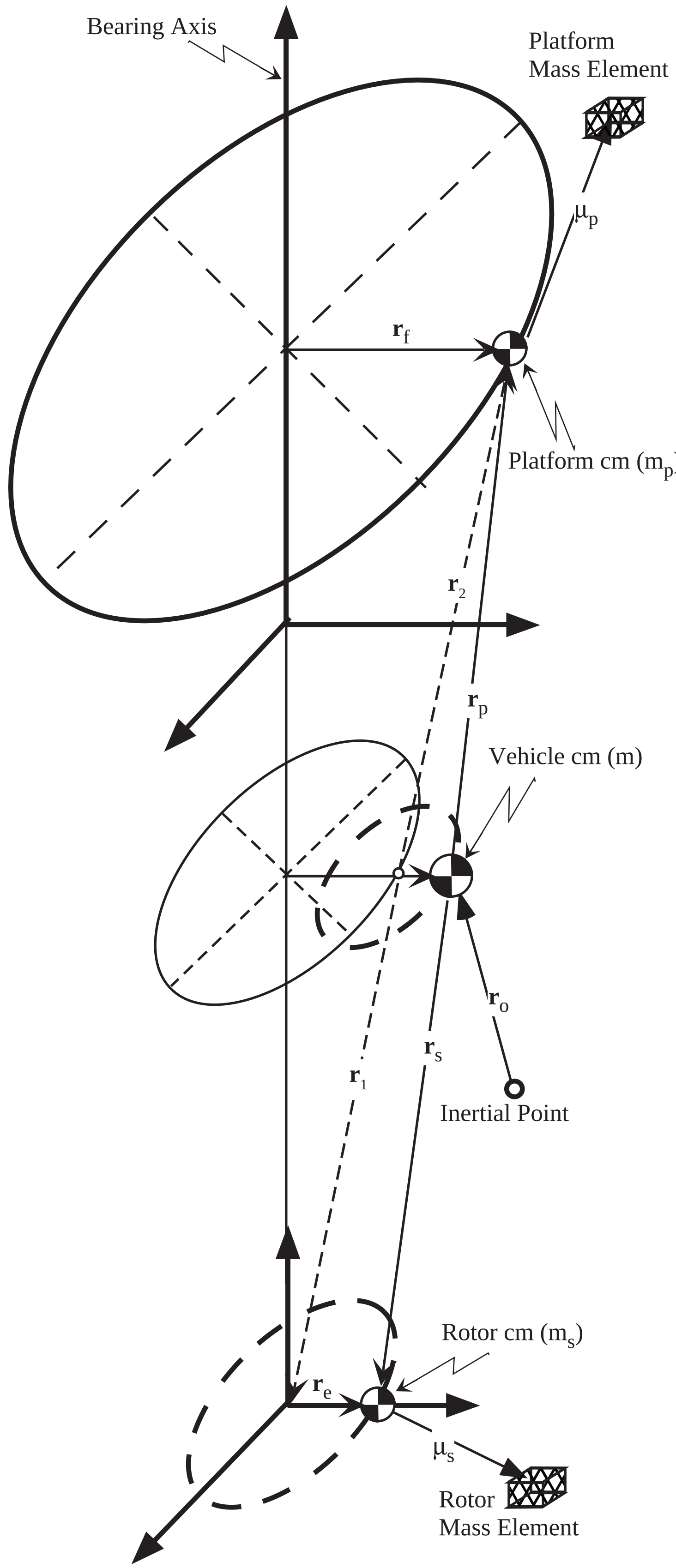


Figure 3.1a Dual-Spin Vehicle Mass Model With Platform and Rotor Static Imbalance.

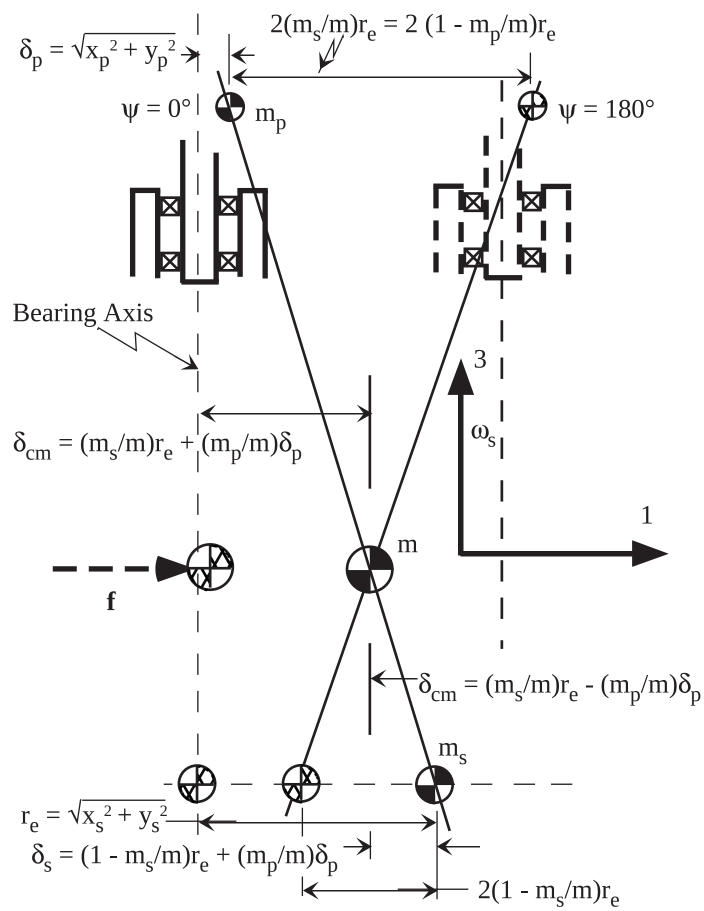


Figure 3.1b Bearing Axis and cm Motion Geometry.

The angular momentum then may be written for the platform and rotor respectively as (these are identical to Eqs. 1.3a and b)

$$\mathbf{H}_p = \int_{\mathbf{P}} (\mathbf{r}_o + \mathbf{r}_p + \boldsymbol{\mu}_p)\times(\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_p + \dot{\boldsymbol{\mu}}_p)dm$$

$$= m_p(\mathbf{r}_o + \mathbf{r}_p)\times(\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_p) + \int \boldsymbol{\mu}_p \times \dot{\boldsymbol{\mu}}_p dm \tag{3.25}$$

$$= m_p(\mathbf{r}_o + \mathbf{r}_p)\times(\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_p) + \mathbf{J}_p \cdot \boldsymbol{\omega}_p$$

and

$$\mathbf{H}_s = \int_{\mathbf{R}} (\mathbf{r}_o + \mathbf{r}_s + \boldsymbol{\mu}_s)\times(\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_s + \dot{\boldsymbol{\mu}}_s)dm.$$

$$= m_s(\mathbf{r}_o + \mathbf{r}_s)\times(\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_s) + \int \boldsymbol{\mu}_s \times \dot{\boldsymbol{\mu}}_s dm \tag{3.26}$$

$$= m_s(\mathbf{r}_o + \mathbf{r}_s)\times(\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_s) + \mathbf{J}_s \cdot \boldsymbol{\omega}_s$$

where

$$\dot{\mathbf{r}}_s = \dot{\mathbf{r}}_1 + (1 - m_s/m)\dot{\mathbf{r}}_e = \boldsymbol{\omega}_p \times \mathbf{r}_1 + (1 - m_s/m)\boldsymbol{\omega}_s \times \mathbf{r}_e \tag{3.27}$$

$$\dot{\mathbf{r}}_p = \dot{\mathbf{r}}_2 - (1 - m_p/m)\dot{\mathbf{r}}_e = \boldsymbol{\omega}_p \times \mathbf{r}_2 - (1 - m_p/m)\boldsymbol{\omega}_s \times \mathbf{r}_e \tag{3.28}$$

Expanding the inertial time derivative of momentum yields

$$\dot{\mathbf{H}}_p = \int_{\mathbf{P}} (\mathbf{r}_o + \mathbf{r}_p + \boldsymbol{\mu}_p)\times(\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_p + \ddot{\boldsymbol{\mu}}_p)dm \tag{3.29}$$

$$= m_p(\mathbf{r}_o + \mathbf{r}_p)\times(\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_p) + \mathbf{J}_p \cdot \dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p \times [\mathbf{J}_p \cdot \boldsymbol{\omega}_p]$$

$$= m_p[\mathbf{r}_o \times \ddot{\mathbf{r}}_o - (1 - m_p/m)\mathbf{r}_o \times \ddot{\mathbf{r}}_e - (1 - m_p/m)\mathbf{r}_e \times \ddot{\mathbf{r}}_o + \mathbf{r}_o \times \ddot{\mathbf{r}}_2 + \mathbf{r}_2 \times \ddot{\mathbf{r}}_o]$$

$$- m_p(1 - m_p/m)[-(1 - m_p/m)\mathbf{r}_e \times \ddot{\mathbf{r}}_e + \mathbf{r}_e \times \ddot{\mathbf{r}}_2 + \mathbf{r}_2 \times \ddot{\mathbf{r}}_e]$$

$$+ \int_{\mathbf{P}} (\mathbf{r}_2 + \boldsymbol{\mu}_p)\times(\ddot{\mathbf{r}}_2 + \ddot{\boldsymbol{\mu}}_p)dm$$

$$\dot{\mathbf{H}}_s = \int_{\mathbf{R}} (\mathbf{r}_o + \mathbf{r}_s + \boldsymbol{\mu}_s)\times(\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_s + \ddot{\boldsymbol{\mu}}_s)dm \tag{3.30}$$

$$= m_s(\mathbf{r}_o + \mathbf{r}_s)\times(\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_s) + \mathbf{J}_s \cdot \dot{\boldsymbol{\omega}}_s + \boldsymbol{\omega}_s \times [\mathbf{J}_s \cdot \boldsymbol{\omega}_s]$$

$$= m_s[\mathbf{r}_o \times \ddot{\mathbf{r}}_o + (1 - m_s/m)\mathbf{r}_o \times \ddot{\mathbf{r}}_e + (1 - m_s/m)\mathbf{r}_e \times \ddot{\mathbf{r}}_o + \mathbf{r}_o \times \ddot{\mathbf{r}}_1 + \mathbf{r}_1 \times \ddot{\mathbf{r}}_o]$$

$$+ m_s(1 - m_s/m)[(1 - m_s/m)\mathbf{r}_e \times \ddot{\mathbf{r}}_e + \mathbf{r}_e \times \ddot{\mathbf{r}}_1 + \mathbf{r}_1 \times \ddot{\mathbf{r}}_e]$$

$$+ \int_{\mathbf{R}} (\mathbf{r}_1 + \boldsymbol{\mu}_s)\times(\ddot{\mathbf{r}}_1 + \ddot{\boldsymbol{\mu}}_s)dm$$

In equation (3.30)

$$\ddot{\mathbf{r}}_s = \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_1 + \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_p \times \mathbf{r}_1) + (1 - m_s/m)[\dot{\boldsymbol{\omega}}_s \times \mathbf{r}_e + \boldsymbol{\omega}_s \times (\boldsymbol{\omega}_s \times \mathbf{r}_e)]$$

$$= \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_1 + \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_p \times \mathbf{r}_1) \tag{3.31}$$

$$+ (1 - m_s/m)[\dot{\boldsymbol{\omega}}_p \times \mathbf{r}_e + \frac{dp\boldsymbol{\omega}_r}{dt} \times \mathbf{r}_e + (\boldsymbol{\omega}_p \times \boldsymbol{\omega}_r) \times \mathbf{r}_e + \boldsymbol{\omega}_s \times (\boldsymbol{\omega}_s \times \mathbf{r}_e)]$$

$$= \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_e + (1 - m_s/m)\,\frac{dp\boldsymbol{\omega}_r}{dt} \times \mathbf{r}_e + \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_p \times \mathbf{r}_1)$$

$$+ (1 - m_s/m)[(\boldsymbol{\omega}_p \times \boldsymbol{\omega}_r) \times \mathbf{r}_e + \boldsymbol{\omega}_s \times (\boldsymbol{\omega}_s \times \mathbf{r}_e)]$$

and similarly for $\ddot{\mathbf{r}}_p$ in (3.29). The last term in each of (3.29) and (3.30) is completely expanded as Eq. 2.22 with inertias as modified by Eqs. 3.11 and 3.12. Note carefully that J_s in the discussion just above Eq. 3.11 is the rotor inertia about the statically balanced rotor cm (tip of $\mathbf{r}_1$ in Figure 3.1), while I_p is the platform inertia about the vehicle cm point with the rotor statically balanced. J_i is used here for body inertia about body cm.

Taking $\mathbf{r}_o = 0$ for the present, we need only expand the terms in $\mathbf{r}_e$ to get the additional torque contributions due to rotor static imbalance. The torque term containing $\mathbf{r}_e$ in (3.29) and (3.30) are respectively denoted $\dot{\mathbf{h}}_s^e$ and $\dot{\mathbf{h}}_p^e$, and summed to get

$$\dot{\mathbf{h}}^e = \dot{\mathbf{h}}_s^e + \dot{\mathbf{h}}_p^e = (m_s m_p/m)\mathbf{r}_e \times \ddot{\mathbf{r}}_e + m_s[\mathbf{r}_e \times \ddot{\mathbf{r}}_1 + \mathbf{r}_1 \times \ddot{\mathbf{r}}_e]\ . \tag{3.32}$$

We require $\ddot{\mathbf{r}}_e$ and $\ddot{\mathbf{r}}_1$. First, transforming $\mathbf{r}_e$ to the platform basis $\mathbf{e}_p$,

$$\mathbf{r}_e = \mathbf{e}_p^T[x_e, y_e, 0]^T = \mathbf{e}_p^T[x_s\cos\psi - y_s\sin\psi,\ x_s\sin\psi + y_s\cos\psi,\ 0]^T \tag{3.33}$$

$$= \mathbf{e}_p^T\sqrt{x_s^2 + y_s^2}[\cos\{\psi + \mathrm{Tan}^{-1}(y_s/x_s)\},\ \sin\{\psi + \mathrm{Tan}^{-1}(y_s/x_s)\},\ 0]^T\ .$$

Denoting the time derivatives of $\mathbf{r}_e$ in $\mathbf{e}_p$ as $\frac{dp\mathbf{r}_e}{dt}$, $\frac{dp2\mathbf{r}_e}{dt^2}$, we get

$$\frac{dp\mathbf{r}_e}{dt} = \mathbf{e}_p^T\dot{\psi}[-y_e, x_e, 0]^T \tag{3.34}$$

$$\frac{dp2\mathbf{r}_e}{dt^2} = \mathbf{e}_p^T[-\dot{\psi}^2 x_e - \ddot{\psi}y_e,\ -\dot{\psi}^2 y_e + \ddot{\psi}x_e,\ 0]^T\ . \tag{3.35}$$

Using

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T[\omega_1, \omega_2, \omega_3]^T\ , \tag{3.36}$$

$$\ddot{\mathbf{r}}_e = \frac{dp2\mathbf{r}_e}{dt^2} + \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_e + 2\boldsymbol{\omega}_p \times \frac{dp\mathbf{r}_e}{dt} + \boldsymbol{\omega}_p \times [\boldsymbol{\omega}_p \times \mathbf{r}_e] \tag{3.37}$$

$$= \mathbf{e}_p^T\begin{bmatrix} -(\dot{\omega}_3 - \omega_1\omega_2 + \ddot{\psi})y_e - (\omega_2^2 + \omega_3^2 + \dot{\psi}^2 + 2\omega_3\dot{\psi})x_e \\ (\dot{\omega}_3 + \omega_1\omega_2 + \ddot{\psi})x_e - (\omega_1^2 + \omega_3^2 + \dot{\psi}^2 + 2\omega_3\dot{\psi})y_e \\ (\dot{\omega}_1 + \omega_2\omega_3 + \omega_2\dot{\psi})y_e - (\dot{\omega}_2 - \omega_1\omega_3 - \omega_1\dot{\psi})x_e \end{bmatrix}.$$

Since $\mathbf{r}_1$ is fixed in $\mathbf{e}_p$,

$$\ddot{\mathbf{r}}_1 = \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_1 + \boldsymbol{\omega}_p \times [\boldsymbol{\omega}_p \times \mathbf{r}_1] \tag{3.38}$$

$$= \mathbf{e}_p^T\begin{bmatrix} (\dot{\omega}_2 + \omega_1\omega_3)z_s + (1 - m_s/m)(\dot{\omega}_3 - \omega_1\omega_2)y_p + (1 - m_s/m)(\omega_2^2 + \omega_3^2)x_p \\ -(\dot{\omega}_1 - \omega_2\omega_3)z_s - (1 - m_s/m)(\dot{\omega}_3 + \omega_1\omega_2)x_p + (1 - m_s/m)(\omega_1^2 + \omega_3^2)y_p \\ -(1 - m_s/m)(\dot{\omega}_1 + \omega_2\omega_3)y_p + (1 - m_s/m)(\dot{\omega}_2 - \omega_1\omega_3)x_p - (\omega_1^2 + \omega_2^2)z_s \end{bmatrix}.$$

Next (3.37) and (3.38) are linearized about the operating point $\boldsymbol{\omega}_p = 0$, and $\dot{\psi} = \omega_s$ (to linearize with the platform spinning is considerably more complex). The accelerations reduce to

$$\ddot{\mathbf{r}}_e = \mathbf{e}_p^T \begin{bmatrix} -y_e\dot{\omega}_s - x_e\omega_s^2 \\ x_e\dot{\omega}_s - y_e\omega_s^2 \\ (\dot{\omega}_1 + \omega_s\omega_2)y_e - (\dot{\omega}_2 - \omega_s\omega_1)x_e \end{bmatrix} \tag{3.39}$$

$$\ddot{\mathbf{r}}_1 = \mathbf{e}_p^T \begin{bmatrix} z_s\dot{\omega}_2 + y_p(1 - m_s/m)\dot{\omega}_3 \\ -z_s\dot{\omega}_1 - x_p(1 - m_s/m)\dot{\omega}_3 \\ -(1 - m_s/m)[y_p\dot{\omega}_1 - x_p\dot{\omega}_2] \end{bmatrix} \tag{3.40}$$

Then expanding (3.32)

$$\dot{\mathbf{h}}^e = \mathbf{e}_p^T \begin{bmatrix} m_s(m_p/m)[(y_e^2 - 2y_ey_p)\dot{\omega}_1 - (x_ey_e - y_ex_p - x_ey_p)\dot{\omega}_2 \\ \quad + (x_ey_e - x_ey_p)\omega_s\omega_1 + (y_e^2 - y_ey_p)\omega_s\omega_2] - m_s[x_ez_s\dot{\omega}_s - y_ez_s\omega_s^2] \\ - - - - - - - - - - - - - - - - - - \\ m_s(m_p/m)[(x_ey_e + x_ey_p + y_ex_p)\dot{\omega}_1 - (x_e^2 + 2x_ex_p)\dot{\omega}_2 \\ \quad + (x_e^2 + x_ex_p)\omega_s\omega_1 + (x_ey_e + y_ex_p)\omega_s\omega_2] - m_s[y_ez_s\dot{\omega}_s + x_ez_s\omega_s^2] \\ - - - - - - - - - - - - - - - - - - \\ -m_s[x_ez_s\dot{\omega}_1 + y_ez_s\dot{\omega}_2] - m_s(m_p/m)[(x_ex_p + y_ey_p)\dot{\omega}_3 \\ \quad - (x_e^2 + y_e^2 - x_ex_p - y_ey_p)\dot{\omega}_s + (x_ey_p - y_ex_p)\omega_s^2] \end{bmatrix} \tag{3.41}$$

These terms (or their unlinearized equivalent) subtract from the right side of (2.22a - c) in the presence of rotor static imbalance given by $\mathbf{r}_e$. To get similar terms for (2.22d) we expand the 3-axis term of $\dot{\mathbf{h}}_s^e$ with the result

$$\dot{h}_3^e = - m_s(m_p/m)[x_ez_s\dot{\omega}_1 + y_ez_s\dot{\omega}_2] + m_s(m_p/m)^2[(x_e^2 + y_e^2 - x_ex_p - y_ey_p)\dot{\omega}_s$$

$$- (x_ex_p + y_ey_p)\dot{\omega}_3 - (x_ey_e - y_ex_p)\omega_s^2] \,. \tag{3.41}$$

Extracting the dominant terms, i.e., proportional to ω_s^2, from (3.40) and (3.41)

$$\dot{\mathbf{h}}^e \approx \mathbf{e}_p^T m_s\omega_s^2 \begin{bmatrix} y_ez_s \\ -x_ez_s \\ -(m_p/m)[x_ey_p - y_ex_p] \end{bmatrix} \tag{3.42}$$

and

$$\begin{aligned} \dot{h}_3^e &\approx - m_s(m_p/m)^2[x_ey_p - y_ex_p]\omega_s^2 \\ &= m_s(m_p/m)(1 - m_p/m)[x_ey_p - y_ex_p]\omega_s^2 - m_s(m_p/m)[x_ey_p - y_ex_p]\omega_s^2 \\ &= D_1 + D_3 \,. \end{aligned} \tag{3.43}$$

If one substitutes x_e and y_e form (3.33) in the 1 and 2-axis terms of (3.42) and compares with the seventh term of Eqs. 2.22a and b, it is observed that rotor static imbalance terms $m_sx_sz_s$ and $m_sy_sz_s$ behave, to first-order, identically as the respective dynamic imbalance terms I_{13}^s and I_{23}^s. Now consider the 3-axis term of (3.42). When rotor and platform momentum derivatives are summed to get this term, internal torques cancel and external torques remain. Therefore, this term in (3.42) can be interpreted as an equivalent spin frequency external torque on the rotor, i.e., an input D_3 on Figure 2.3. Then the term of (3.43) can be interpreted as an external torque (D_3) plus an internal torque equivalent to D_1 of Figure 2.3. Thus, the despin pointing effect of rotor static imbalance can be obtained to first-order by multiplying these two torques by the respective closed-loop disturbance transmission functions. Note the despin disturbance effect vanishes as expected if the platform is statically balanced.

One can further simplify the torque terms of (3.43) by letting

$$\mathbf{r}_f = \mathbf{e}_p^T[x_f, y_f, 0]^T \tag{3.44}$$

be the platform cm offset from the bearing axis. Then using $\mathbf{r}_e$ from (3.33) and substituting for x_p, y_p,

$$D_1 = m_s(m_p/m)(1 - m_p/m)^2\sqrt{(x_s^2 + y_s^2)(x_f^2 + y_f^2)}\,\omega_s^2\cos(\psi + \psi_1) \tag{3.45}$$

$$D_3 = D_1/(1 - m_p/m) \,, \tag{3.46}$$

where

$$\psi_1 = \mathrm{Tan}^{-1}\{(y_s y_f + x_s x_f)/(x_s y_f - y_s x_f)\} = \mathrm{Tan}^{-1}\{y_s/x_s\} + \mathrm{Tan}^{-1}\{x_f/y_f\} \ . \tag{3.47}$$

3.3 Multiple Statically Balanced Rotors

A simple extension of the derivation of Section 3.1 allows us to admit multiple statically balanced rotors with bearing axes coaligned along the 3-axis. It is not necessary that the bearing axes be colocated so long as they are coaligned. We call the new bodies "rotors" to distinguish them from the one "platform" body that need not be statically balanced, although the labeling is somewhat arbitrary. However one or more of the new bodies can be despun or nearly so, as for example the despun solar array of the three body HS394 spacecraft developed at Hughes in the mid 80's.

Noting that under addition of more statically balanced rotors (cm on the bearing axis of each respective rotor) the total vehicle cm remains fixed in the platform as each rotor turns. Hence the torque equations for each of i rotors become (analogous to Eq. 3.5)

$$\dot{\mathbf{H}}_{s_i} = \mathbf{J}_{s_i}\cdot\dot{\boldsymbol{\omega}}_{s_i} + \boldsymbol{\omega}_{s_i}\times[\mathbf{J}_{s_i}\cdot\boldsymbol{\omega}_{s_i}] - m_{s_i}\mathbf{r}_{s_i}\times[\mathbf{r}_{s_i}\times\dot{\boldsymbol{\omega}}_p] - m_{s_i}\mathbf{r}_{s_i}\times[\boldsymbol{\omega}_p\times(\mathbf{r}_{s_i}\times\boldsymbol{\omega}_p)] \ . \tag{3.48}$$

The derivative of total momentum becomes, similar to Eq. 3.6

$$\dot{\mathbf{H}} = \sum_i\left[\mathbf{J}_{s_i}\cdot\dot{\boldsymbol{\omega}}_{s_i} + \boldsymbol{\omega}_{s_i}\times[\mathbf{J}_{s_i}\cdot\boldsymbol{\omega}_{s_i}]\right] + \hat{\mathbf{I}}_p\cdot\dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p\times[\hat{\mathbf{I}}_p\cdot\boldsymbol{\omega}_p] \ . \tag{3.49}$$

In this case $\hat{\mathbf{I}}_p$ has a correction term for each rotor, becoming

$$\hat{\mathbf{I}}_p = \mathbf{e}_p^T\,\hat{I}_p\,\mathbf{e}_p \ . = \mathbf{e}_p^T[I_p - \sum_i m_{s_i}\tilde{r}_{s_i}\tilde{r}_{s_i}]\mathbf{e}_p \ , \tag{3.50}$$

where $\hat{I}_p$ is the generalized platform inertia. A parallel to (3.15) applies using $\sum_i J_{s_i}$.

Now we find that Eqs. 2.22a-c extend to multiple rotors by summing in the terms for each, and the total vehicle torque equations are obtained by collecting these three equations along with one scalar equation of the form in 2.22d for each rotor.

Linearized time invariant equations can be obtained in the platform if every rotor can be approximated as balanced and symmetric. These equations will be identical in form to (2.34) with $\hat{I}$ substituted for I and nutation frequency

$$\lambda_1 = [\sum_i I_{33}^{s_i}\omega_{s_i} + (\hat{I}_{33}^{p} - \hat{I}_{22})\omega_p]/I_{11} \tag{3.51a}$$

$$\lambda_2 = [\sum_i I_{33}^{s_i}\omega_{s_i} + (\hat{I}_{33}^{p} - \hat{I}_{11})\omega_p]/I_{22} \tag{3.51b}$$

$$\lambda_p = \sqrt{\lambda_1\lambda_2} \approx H/I_T - \omega_p \tag{3.51c}$$

$$\lambda_o = H/I_T = \sum_i I_{33}^{s_i}\omega_{s_i}/I_T \ \ ; \ \omega_p = 0 \ . \tag{3.51d}$$

To express the equations in any rotor one must also approximate the platform as well as all other rotors as balanced and symmetric. When expressed in rotor i, the approximate equations will have the same form as the simple spinner or a dual-spin vehicle, but having nutation frequency

$$\lambda_{s_i} = H/I_T - \omega_{s_i} = \lambda_o - \omega_{s_i} \ . \tag{3.52}$$

4.0 Spacecraft Acceleration and Moments

4.1 Acceleration of a Point on the Rotor

Let $\mathbf{r}_o$ be the inertial position of the vehicle cm and $\mathbf{r}_a$ be the position of a rotor fixed point with respect to the the cm. Then denoting

$$\mathbf{R} = \mathbf{r}_o + \mathbf{r}_a \ , \tag{4.1}$$

the second inertial time derivative is

$$\ddot{\mathbf{R}} = \ddot{\mathbf{r}}_o + \frac{{}^s d^2\mathbf{r}_a}{dt^2} + \dot{\boldsymbol{\omega}}_s \times \mathbf{r}_a + 2\boldsymbol{\omega}_s \times \frac{{}^s d\mathbf{r}_a}{dt} + \boldsymbol{\omega}_s \times [\boldsymbol{\omega}_s \times \mathbf{r}_a] \ , \tag{4.2}$$

where

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T[\omega_{s1}, \omega_{s2}, \omega_{s3}]^T \tag{4.3}$$

is the inertial angular rate of the rotor basis $\mathbf{e}_s$ and $\frac{{}^s d\mathbf{r}_a}{dt}$ denotes time differentiation in this basis. Using

$$\begin{aligned} \mathbf{r}_a = \boldsymbol{\delta} + \mathbf{r}_1 &= \mathbf{e}_p^T[\delta_1, \delta_2, 0]^T + \mathbf{e}_s^T[r_1, r_2, r_3]^T \\ &= \mathbf{e}_s^T[\delta_1 \cos\psi + \delta_2 \sin\psi + r_1,\ \delta_2 \cos\psi - \delta_1 \sin\psi + r_2,\ r_3]^T \\ &= \mathbf{e}_s^T[\delta_{s1} + r_1,\ \delta_{s2} + r_2,\ r_3]^T \end{aligned} \tag{4.4}$$

and expanding

$$\begin{aligned} \ddot{\mathbf{R}} &= \ddot{\mathbf{r}}_o + \frac{{}^s d^2\mathbf{r}_a}{dt^2} + \dot{\boldsymbol{\omega}}_s \times \boldsymbol{\delta} + 2\boldsymbol{\omega}_s \times \frac{{}^s d\mathbf{r}_a}{dt} + \boldsymbol{\omega}_s \times [\boldsymbol{\omega}_s \times \boldsymbol{\delta}] \\ &+ \mathbf{e}_s^T \begin{bmatrix} \dot{\omega}_{s2} r_3 - \dot{\omega}_{s3} r_2 + \omega_{s1}\omega_{s2} r_2 + \omega_{s1}\omega_{s3} r_3 - [\omega_{s2}^2 + \omega_{s3}^2] r_1 \\ \dot{\omega}_{s3} r_1 - \dot{\omega}_{s1} r_3 + \omega_{s1}\omega_{s2} r_1 + \omega_{s2}\omega_{s3} r_3 - [\omega_{s1}^2 + \omega_{s3}^2] r_2 \\ \dot{\omega}_{s1} r_2 - \dot{\omega}_{s2} r_1 + \omega_{s1}\omega_{s3} r_1 + \omega_{s2}\omega_{s3} r_2 - [\omega_{s1}^2 + \omega_{s2}^2] r_3 \end{bmatrix} \end{aligned} \tag{4.5}$$

where we have taken $\mathbf{r}_1$ fixed in $\mathbf{e}_s$ and the unexpanded terms, excepting $\ddot{\mathbf{r}}_o$, are present only in the case of a platform static imbalance resulting in vehicle cm offset $\boldsymbol{\delta}$. The effect of rotor static imbalance is fixed in the rotor and included in $\mathbf{r}_1$ if present.

It is sometimes convenient to express $\ddot{\mathbf{R}}$ in terms of platform inertial rates in the platform basis $\mathbf{e}_p$ instead of rotor rates as above, i.e.,

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T[\omega_{p1}, \omega_{p2}, \omega_{p3}]^T \ . \tag{4.6}$$

The relative rate is

$$\boldsymbol{\omega}_r = \mathbf{e}_p^T[0, 0, \dot{\psi}]^T = \mathbf{e}_s^T[0, 0, \dot{\psi}]^T \ , \tag{4.7}$$

where ψ ($\omega_r t$ for constant rotor rate) is the relative phase. The rotor rate is then

$$\boldsymbol{\omega}_s = \boldsymbol{\omega}_p + \boldsymbol{\omega}_r = \mathbf{e}_s^T \begin{bmatrix} \omega_{s1} \\ \omega_{s2} \\ \omega_{s3} \end{bmatrix} = \mathbf{e}_s^T \begin{bmatrix} \omega_{p1}\cos\psi + \omega_{p2}\sin\psi \\ \omega_{p2}\cos\psi - \omega_{p1}\sin\psi \\ \omega_{p3} + \dot{\psi} \end{bmatrix} . \tag{4.8}$$

The inertial derivative is

$$\dot{\boldsymbol{\omega}}_s = \dot{\boldsymbol{\omega}}_p + \frac{{}^s d\boldsymbol{\omega}_r}{dt} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_r$$

$$= \mathbf{e}_s^T \begin{bmatrix} \dot{\omega}_{s1} \\ \dot{\omega}_{s2} \\ \dot{\omega}_{s3} \end{bmatrix} = \mathbf{e}_s^T \begin{bmatrix} (\dot{\omega}_{p1} + \dot{\psi}\omega_{p2}) \cos\psi + (\dot{\omega}_{p2} - \dot{\psi}\omega_{p1}) \sin\psi \\ (\dot{\omega}_{p2} - \dot{\psi}\omega_{p1}) \cos\psi - (\dot{\omega}_{p1} + \dot{\psi}\omega_{p2}) \sin\psi \\ \dot{\omega}_{p3} + \ddot{\psi} \end{bmatrix}, \tag{4.9}$$

and substitution of (4.8) and (4.9) in (4.5) yields the acceleration of a rotor point in terms of platform angular rates and accelerations. Equation 4.5 is valid for a point either in the rotor or platform provided the angular rate of the appropriate body is substituted and $\mathbf{r}_a$ is with respect to the correct basis. A solution frequently used is the acceleration due to small angle sinusoidal motions. Let $\ddot{\mathbf{r}}_o = {}^s d\mathbf{r}_a/dt = 0$, and

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T [\omega_o \cos\lambda_s t,\ \omega_o \sin\lambda_s t,\ \omega_3]^T \ . \tag{4.10}$$

Then the first-order acceleration is

$$\ddot{\mathbf{R}} \approx \ddot{\mathbf{r}}_a \approx \mathbf{e}_s^T \begin{bmatrix} \omega_o(\lambda_s + \omega_3) r_3 \cos\lambda_s t - \omega_3^2 r_1 \\ \omega_o(\lambda_s + \omega_3) r_3 \sin\lambda_s t - \omega_3^2 r_2 \\ \omega_o(\omega_3 - \lambda_s)\{r_1 \cos\lambda_s t + r_2 \sin\lambda_s t\} \end{bmatrix}. \tag{4.11a}$$

For nutation frequency $\lambda_s = (\sigma - 1)\omega_s$, and $\omega_3 = \omega_s$, yielding

$$\ddot{\mathbf{R}} \approx \ddot{\mathbf{r}}_a \approx \mathbf{e}_s^T \left\{ \omega_o \omega_s \begin{bmatrix} \sigma r_3 \cos\lambda_s t \\ \sigma r_3 \sin\lambda_s t \\ (2 - \sigma)\{r_1 \cos\lambda_s t + r_2 \sin\lambda_s t\} \end{bmatrix} - \omega_s^2 \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} \right\}, \tag{4.11b}$$

where $\omega_o = \sigma\omega_s\theta_n$ is the nutation transverse rate magnitude. For large angle nutation of magnitude $\tan\theta_n = \sqrt{H_1^2 + H_2^2}/H_3$, the axial acceleration on a symmetric vehicle is found from 4.5 as

$$a_3 = \sigma(2 - \sigma)\omega_s^2 \tan\theta_n \sqrt{r_1^2 + r_2^2} \cos(\lambda_s t - \mathrm{Tan}^{-1} r_2/r_1) - \sigma^2 \omega_s^2 r_3 \tan^2\theta_n \ . \tag{4.11c}$$

Now we return to expand the platform static imbalance terms from Eq. 4.5, viz.,

$$2\boldsymbol{\omega}_s \times \frac{{}^s d\mathbf{r_a}}{dt} = \mathbf{e}_s^T 2 \begin{bmatrix} \omega_{s2}\dot{\delta}_{s3} - \omega_{s3}\dot{\delta}_{s2} \\ \omega_{s3}\dot{\delta}_{s1} - \omega_{s1}\dot{\delta}_{s3} \\ \omega_{s1}\dot{\delta}_{s2} - \omega_{s2}\dot{\delta}_{s1} \end{bmatrix}$$

$$= \mathbf{e}_s^T 2\dot{\psi} \begin{bmatrix} \omega_{s3}[\delta_1 \cos\psi + \delta_2 \sin\psi] \\ \omega_{s3}[\delta_2 \cos\psi - \delta_1 \sin\psi] \\ -\omega_{s1}[\delta_1 \cos\psi + \delta_2 \sin\psi] - \omega_{s2}[\delta_2 \cos\psi - \delta_1 \sin\psi] \end{bmatrix}, \tag{4.12}$$

$$\frac{{}^s d^2\mathbf{r_a}}{dt^2} = \mathbf{e}_s^T \ddot{\psi} [\delta_2 \cos\psi - \delta_1 \sin\psi,\ -\delta_1 \cos\psi - \delta_2 \sin\psi,\ 0]^T$$

$$+ \mathbf{e}_s^T \dot{\psi}^2 [-\delta_1 \cos\psi - \delta_2 \sin\psi,\ \delta_1 \sin\psi - \delta_2 \cos\psi,\ 0]^T \ , \tag{4.13}$$

$$\boldsymbol{\omega}_s \times [\boldsymbol{\omega}_s \times \boldsymbol{\delta}] = \mathbf{e}_s^T \begin{bmatrix} \omega_{s1}\omega_{s2}\delta_{s2} - [\omega_{s2}^2 + \omega_{s3}^2]\delta_{s1} \\ \omega_{s1}\omega_{s2}\delta_{s1} - [\omega_{s1}^2 + \omega_{s3}^2]\delta_{s2} \\ \omega_{s1}\omega_{s3}\delta_{s1} + \omega_{s2}\omega_{s3}\delta_{s2} \end{bmatrix}$$

$$= \mathbf{e}_s^T \begin{bmatrix} \omega_{s1}\omega_{s2}[\delta_2 \cos\psi - \delta_1 \sin\psi] - [\omega_{s2}^2 + \omega_{s3}^2][\delta_1 \cos\psi + \delta_2 \sin\psi] \\ \omega_{s1}\omega_{s2}[\delta_1 \cos\psi + \delta_2 \sin\psi] - [\omega_{s1}^2 + \omega_{s3}^2][\delta_2 \cos\psi - \delta_1 \sin\psi] \\ \omega_{s1}\omega_{s3}[\delta_1 \cos\psi + \delta_2 \sin\psi] + \omega_{s2}\omega_{s3}[\delta_2 \cos\psi - \delta_1 \sin\psi] \end{bmatrix}, \tag{4.14}$$

and

$$\dot{\boldsymbol{\omega}}_s \times \boldsymbol{\delta} = \mathbf{e}_s^T \begin{bmatrix} -\dot{\omega}_{s3}[\delta_2 \cos\psi - \delta_1 \sin\psi] \\ \dot{\omega}_{s3}[\delta_1 \cos\psi + \delta_2 \sin\psi] \\ \dot{\omega}_{s1}[\delta_2 \cos\psi - \delta_1 \sin\psi] - \dot{\omega}_{s2}[\delta_1 \cos\psi + \delta_2 \sin\psi] \end{bmatrix}. \tag{4.15}$$

The preceding four components may be collected to get the total acceleration due to platform static imbalance $\boldsymbol{\delta}$. Doing so for the 3-axis (spin) only and using the symbol ∂a_3 to denote the additive acceleration component induced by platform static imbalance, we obtain

$$\partial a_3 = -(2\dot{\psi} - \omega_{s3})\{[\omega_{s1}\delta_1 + \omega_{s2}\delta_2]\cos\psi + [\omega_{s1}\delta_2 - \omega_{s2}\delta_1]\sin\psi\}$$

$$+ \dot{\omega}_{s1}[\delta_2 \cos\psi - \delta_1 \sin\psi] - \dot{\omega}_{s2}[\delta_1 \cos\psi + \delta_2 \sin\psi] \,. \tag{4.16}$$

Assuming a balanced symmetric platform, while admitting rotor asymmetry, the nutation rates

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T[\omega_o \cos\lambda_s t,\ \eta\omega_o \sin\lambda_s t,\ \omega_s]^T\ , \tag{4.17}$$

obtain, where the factor

$$\eta = \sqrt{[I_{11}(I_{33}^s - I_{11})]/[I_{22}(I_{33}^s - I_{22})]} \tag{4.18}$$

arises from rotor asymmetry(see Appendix C). The axial acceleration component is

$$\partial a_3 = -(2\dot{\psi} - \omega_s)\omega_o\{[\delta_1 \cos\lambda_s t + \eta\delta_2 \sin\lambda_s t]\cos\psi + [\delta_2 \cos\lambda_s t - \eta\delta_1 \sin\lambda_s t]\sin\psi\} \tag{4.19}$$

$$- \omega_o\lambda_s \sin\lambda_s t[\delta_2 \cos\psi - \delta_1 \sin\psi] - \eta\omega_o\lambda_s \cos\lambda_s t[\delta_1 \cos\psi + \delta_2 \sin\psi] \,.$$

$$= -(2\dot{\psi} - \omega_s)\omega_o\{[\delta_1\cos(\lambda_s t + \psi) + \delta_2\sin(\lambda_s t + \psi)] + (1-\eta)[\delta_1 \sin\lambda_s t \sin\psi - \delta_2 \sin\lambda_s t \cos\psi]\}$$

$$-\omega_o\lambda_s\{[\delta_1\cos(\lambda_s t + \psi) + \delta_2\sin(\lambda_s t + \psi)] - (1-\eta)[\delta_1 \cos\lambda_s t \cos\psi + \delta_2 \cos\lambda_s t \sin\psi]\}$$

$$= -(2\dot{\psi} - \omega_s)\omega_o\{[\delta_1\cos(\lambda_s t + \psi) + \delta_2\sin(\lambda_s t + \psi)] + (1-\eta)[\delta_1 \sin\psi - \delta_2 \cos\psi]\sin\lambda_s t\}$$

$$-\omega_o\lambda_s\{[\delta_1\cos(\lambda_s t + \psi) + \delta_2\sin(\lambda_s t + \psi)] - (1-\eta)[\delta_1 \cos\psi + \delta_2 \sin\psi]\cos\lambda_s t\}$$

$$= -(2\dot{\psi} + \lambda_s - \omega_s)\omega_o\{[\delta_1\cos(\lambda_s t + \psi) + \delta_2\sin(\lambda_s t + \psi)]\}$$

$$- (1-\eta)\omega_o\{(2\dot{\psi} - \omega_s)[\delta_1 \sin\psi - \delta_2 \cos\psi]\sin\lambda_s t - \lambda_s[\delta_1 \cos\psi + \delta_2 \sin\psi]\cos\lambda_s t\}$$

$$= -(2\dot{\psi} + \lambda_s - \omega_s)\omega_o\{[\delta_1\cos(\lambda_s t + \psi) + \delta_2\sin(\lambda_s t + \psi)]\}$$

$$+ (1-\eta)\omega_o\Big[(2\dot{\psi} - \omega_s + \lambda_s)\{\delta_1/2\cos(\lambda_s t + \psi) + \delta_2/2\cos(\lambda_s t + \psi)\}$$

$$- (2\dot{\psi} - \omega_s - \lambda_s)\{\delta_1/2\cos(\lambda_s t - \psi) - \delta_2/2\cos(\lambda_s t - \psi)\}\Big]$$

Thus the combined effects of platform static imbalance and rotor asymmetry produces acceleration at frequencies $\lambda_s \pm \omega_r$, which for a despun platform reduces to $\lambda_o = \sigma\omega_s$ (inertial nutation frequency) and $(2-\sigma)\omega_s$. Magnitude and frequencies are tabulated below.

For an asymmetric platform the transverse rates are derived in Section 5.5 and have the form

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T\omega_o[\nu_1 \cos\{\lambda_s t + \beta_1\} - \nu_2 \cos\{(\lambda_p + \omega_s)t + \beta_2\},\ \nu_1 \sin\{\lambda_s t + \beta_1\} + \nu_2 \sin\{(\lambda_p + \omega_s)t + \beta_2\},\ \omega_s/\omega_o]^T\ . \quad (4.20)$$

Here the platform asymmetry coefficients are $\nu_1 = A/\omega_o = AI_{11}/T_1 \approx (1 + \sqrt{I_{11}/I_{22}})/2$, $\nu_2 = B/\omega_o = BI_{11}/T_1 \approx (1 - \sqrt{I_{11}/I_{22}})/2$, where A, B are developed in Sect 5.5. Expanding the axial acceleration

$$\begin{aligned}
\partial a_3 = &- (2\dot{\psi} - \omega_s)\omega_o\nu_1\Big\{[\delta_1 \cos\{\lambda_s t + \beta_1\} + \delta_2 \sin\{\lambda_s t + \beta_1\}]\cos\psi + [\delta_2 \cos\{\lambda_s t + \beta_1\} - \delta_1 \sin\{\lambda_s t + \beta_1\}]\sin\psi\Big\} \\
&- \omega_o\lambda_s\nu_1\Big\{\sin\{\lambda_s t + \beta_1\}[\delta_2 \cos\psi - \delta_1 \sin\psi] + \cos\{\lambda_s t + \beta_1\}[\delta_1 \cos\psi + \delta_2 \sin\psi]\Big\} \\
&+ (2\dot{\psi} - \omega_s)\omega_o\nu_2\Big\{[\delta_1 \cos\{(\lambda_p + \omega_s)t + \beta_2\} - \delta_2 \sin\{(\lambda_p + \omega_s)t + \beta_2\}]\cos\psi \\
&\qquad + [\delta_2 \cos\{(\lambda_p + \omega_s)t + \beta_2\} + \delta_1 \sin\{(\lambda_p + \omega_s)t + \beta_2\}]\sin\psi\Big\} \\
&+ (\lambda_p + \omega_s)\omega_o\nu_2\Big\{\sin\{(\lambda_p + \omega_s)t + \beta_2\}[\delta_2 \cos\psi - \delta_1 \sin\psi] - \cos\{(\lambda_p + \omega_s)t + \beta_2\}[\delta_1 \cos\psi + \delta_2 \sin\psi]\Big\} \\
= &- (2\dot{\psi} - \omega_s + \lambda_s)\omega_o\nu_1[\delta_1 \cos\{\lambda_s t + \psi + \beta_1\} + \delta_2 \sin\{\lambda_s t + \psi + \beta_1\}] \qquad (4.21) \\
&+ (2\dot{\psi} - 2\omega_s - \lambda_p)\omega_o\nu_2[\delta_1 \cos\{(\lambda_p + \omega_s)t - \psi + \beta_2\} - \delta_2 \sin\{(\lambda_p + \omega_s)t - \psi + \beta_2\}]
\end{aligned}$$

The resultant frequencies are again tabulated below.

Table 4.1 Axial Acceleration Frequency and Magnitude Induced by Nutation.

Source	Frequency	Magnitude
Nutation	$\lambda_s = \lambda_o - \omega_s = (\sigma - 1)\omega_s$	$\theta_n r_o(\omega_s - \lambda_s)\lambda_o = \theta_n r_o(2 - \sigma)\sigma\omega_s^2$
Rotor Asymmetry	λ_s	$\theta_n[r_1^2(\omega_s - \eta\lambda_s)^2 + r_2^2(\eta\omega_s - \lambda_s)^2]^{1/2}\lambda_o$
Platform Asymmetry	λ_s $\lambda_p + \omega_s$	$\theta_n\nu_1 r_o(\omega_s - \lambda_s)\lambda_o$ $\theta_n\nu_2 r_o(\lambda_p + 2\omega_s)\lambda_o$
Platform Static Imbalance	$\lambda_o = \sigma\omega_s$	$\theta_n\delta\lambda_o^2$
Rotor Asymmetry and Platform Static Imbalance	$\lambda_s + \omega_r$ $\lambda_s - \omega_r$	$\theta_n(\delta/2)(1 - \eta)(2\omega_r - \omega_s + \lambda_s)\lambda_o$ $\theta_n(\delta/2)(1 - \eta)(2\omega_r - \omega_s - \lambda_s)\lambda_o$
Platform Asymmetry and Platform Static Imbalance	$\lambda_s + \omega_r$ $\lambda_p + \omega_s - \omega_r$	$\theta_n\nu_1\delta(2\omega_r - \omega_s + \lambda_s)\lambda_o$ $\theta_n\nu_2\delta(2\omega_r - 2\omega_s + \lambda_p)\lambda_o$

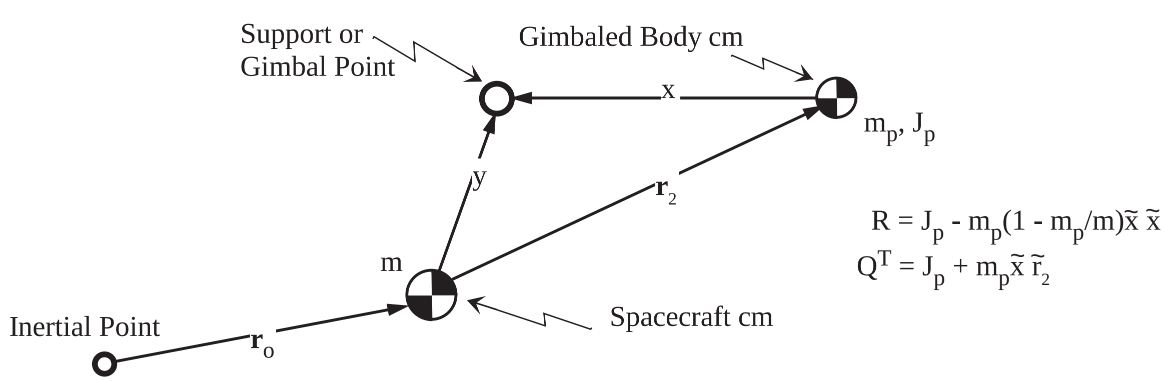


Figure 4.1 Appendage Mass Model.

4.2 Appendage Support Moments (Despin Bearing Bending Moments)

Let $\mathbf{r}_o$ be the position of the vehicle cm with respect to an inertial point, $\mathbf{r}_2$ be the position of the platform cm with respect to the vehicle cm, and $\mathbf{x}$ be the position of the despin bearing center of symmetry (platform support point) with respect to the platform cm. The two bearings apply forces $\mathbf{F}_1$, $\mathbf{F}_2$ to the platform at points $\Delta\mathbf{x}_1$, $\Delta\mathbf{x}_2$ respectively displaced from the symmetry point. Then

$$\dot{\mathbf{H}}_p = \int[\mathbf{r}_o + \mathbf{r}_2 + \mu_p]\times[\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_2 + \ddot{\mu}_p]dm = \int \mu_p\times\ddot{\mu}_p dm + m_p[\mathbf{r}_o + \mathbf{r}_2]\times[\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_2]$$

$$= \int \mu_p\times\ddot{\mu}_p dm + m_p\mathbf{r}_2\times\ddot{\mathbf{r}}_2 + m_p[\mathbf{r}_2\times\ddot{\mathbf{r}}_o + \mathbf{r}_o\times\ddot{\mathbf{r}}_2 + \mathbf{r}_o\times\ddot{\mathbf{r}}_o]$$

$$= \mathbf{J}_p\cdot\dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p\times[\mathbf{J}_p\cdot\omega_p] + m_p\mathbf{r}_2\times\ddot{\mathbf{r}}_2 + m_p[\mathbf{r}_2\times\ddot{\mathbf{r}}_o + \mathbf{r}_o\times\ddot{\mathbf{r}}_2 + \mathbf{r}_o\times\ddot{\mathbf{r}}_o]\ , \qquad (4.22)$$

and if $\mathbf{r}_2$ is fixed in the platform basis $\mathbf{e}_p$ (rotor statically balanced), $\dot{\mathbf{H}}_p$ simplifies to

$$\dot{\mathbf{H}}_p = \mathbf{I}_p\cdot\dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p\times[\mathbf{I}_p\cdot\omega_p] + m_p[\mathbf{r}_2\times\ddot{\mathbf{r}}_o + \mathbf{r}_o\times\ddot{\mathbf{r}}_2 + \mathbf{r}_o\times\ddot{\mathbf{r}}_o]\ . \qquad (4.23)$$

Here we denote inertia dyadics with respect to platform and spacecraft cm respectively as $\mathbf{J}_p$ and $\mathbf{I}_p$. The moments on the platform are

$$\mathbf{M}_p = [\mathbf{r}_o + \mathbf{r}_2 + \mathbf{x} + \Delta\mathbf{x}_1]\times\mathbf{F}_1 + [\mathbf{r}_o + \mathbf{r}_2 + \mathbf{x} + \Delta\mathbf{x}_2]\times\mathbf{F}_2$$

$$= [\mathbf{r}_o + \mathbf{r}_2 + \mathbf{x}]\times[\mathbf{F}_1 + \mathbf{F}_2] + \Delta\mathbf{x}_1\times\mathbf{F}_1 + \Delta\mathbf{x}_2\times\mathbf{F}_2 \qquad (4.24)$$

$$= [\mathbf{r}_o + \mathbf{r}_2 + \mathbf{x}]\times[\mathbf{F}_1 + \mathbf{F}_2] + \mathbf{M}_b\ ,$$

where $\mathbf{M}_b$ contains the bending moments about the 1 and 2-axes as well as the despin torque. Assuming $\mathbf{F}_1$, $\mathbf{F}_2$ are the only forces on the platform, such that $\mathbf{F}_1 + \mathbf{F}_2 = m_p[\ddot{\mathbf{r}}_2 + \ddot{\mathbf{r}}_o]$, and equating $\dot{\mathbf{H}}_p$ to $\mathbf{M}_p$, the bending moments become the transverse components of

$$\mathbf{M}_b = \mathbf{J}_p\cdot\dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p\times[\mathbf{J}_p\cdot\boldsymbol{\omega}_p] - m_p\mathbf{x}\times[\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_2]\ . \qquad (4.25)$$

Although the description refers to support moments for the entire despun platform, the model applies equally well to any appendage. Defining

$$\mathbf{r}_2 = \mathbf{y} - \mathbf{x}\ , \qquad (4.26)$$

the moments may be rewritten

$$\mathbf{M}_b = \mathbf{J}_p\cdot\dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p\times[\mathbf{J}_p\cdot\omega_p] + m_p\mathbf{r}_2\times\ddot{\mathbf{r}}_2 - m_p[\mathbf{y}\times\ddot{\mathbf{r}}_2 + \mathbf{x}\times\ddot{\mathbf{r}}_o]\ , \qquad (4.27)$$

and with $\mathbf{r}_2$ fixed in $\mathbf{e}_p$

$$\mathbf{M}_b = \mathbf{I}_p\cdot\dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p\times[\mathbf{I}_p\cdot\boldsymbol{\omega}_p] - m_p[\mathbf{y}\times\ddot{\mathbf{r}}_2 + \mathbf{x}\times\ddot{\mathbf{r}}_o]\ . \qquad (4.28)$$

Taking

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T[\omega_1, \omega_2, \omega_3]^T \qquad (4.29)$$

$$\mathbf{y} = \mathbf{e}_p^T[y_1, y_2, y_3]^T\ , \qquad (4.30)$$

and $\mathbf{r}_2$ fixed in $\mathbf{e}_p$ as

$$\mathbf{r}_2 = \mathbf{e}_p^T[r_1, r_2, r_3]^T\ , \qquad (4.31)$$

the moment components expand as

$$\begin{aligned} M_1 = &+ [I^p_{11} - m_p(y_2r_2 + y_3r_3)]\dot{\omega}_1 - [I^p_{12} - m_py_2r_1]\dot{\omega}_2 - [I^p_{13} - m_py_3r_1]\dot{\omega}_3 \\ &- [I^p_{13} - m_py_3r_1]\omega_1\omega_2 + [I^p_{12} - m_py_2r_1]\omega_1\omega_3 + [I^p_{33} - I^p_{22} + m_p(y_3r_3 - y_2r_2)]\omega_2\omega_3 \qquad (4.32a) \\ &- [m_p(y_3r_2 - y_2r_3)]\omega_1^2 - [I^p_{23} - m_py_2r_3]\omega_2^2 + [I^p_{23} - m_py_3r_2]\omega_3^2 \end{aligned}$$

$$\begin{aligned} M_2 = &- [I^p_{12} - m_py_1r_2]\dot{\omega}_1 + [I^p_{22} - m_p(y_1r_1 + y_3r_3)]\dot{\omega}_2 - [I^p_{23} - m_py_3r_2]\dot{\omega}_3 \\ &+ [I^p_{23} - m_py_3r_2]\omega_1\omega_2 + [I^p_{11} - I^p_{33} + m_p(y_1r_1 - y_3r_3)]\omega_1\omega_3 - [I^p_{12} - m_py_1r_2]\omega_2\omega_3 \qquad (4.32b) \end{aligned}$$

$$+ [I_{13}^p - m_p y_1 r_3]\omega_1^2 + [m_p(y_3 r_1 - y_1 r_3)]\omega_2^2 - [I_{13}^p - m_p y_3 r_1]\omega_3^2$$

$$\begin{aligned} M_3 = & -[I_{13}^p - m_p y_1 r_3]\dot\omega_1 - [I_{23}^p - m_p y_2 r_3]\dot\omega_2 + [I_{33}^p - m_p(y_1 r_1 + y_2 r_2)]\dot\omega_3 \\ & + [I_{22}^p - I_{11}^p + m_p(y_2 r_2 - y_1 r_1)]\omega_1\omega_2 - [I_{23}^p - m_p y_2 r_3]\omega_1\omega_3 + [I_{13}^p - m_p y_1 r_3]\omega_2\omega_3 \\ & - [I_{12}^p - m_p y_1 r_2]\omega_1^2 + [I_{12}^p - m_p y_2 r_1]\omega_2^2 + [m_p(y_1 r_2 - y_2 r_1)]\omega_3^2 \end{aligned} \quad (4.32c)$$

Note that the expansion can be evaluated with elements of $\mathbf{J}_p$ by substituting J_{ij}^p for I_{ij}^p and x_i for y_i.

For small motion of a despun platform appendage elimination of second-order terms yields $\ddot{\mathbf{r}}_2 \approx \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_2$ and

$$\begin{aligned} \mathbf{M}_b \approx \mathbf{I}_p \cdot \dot{\boldsymbol{\omega}}_p - m_p \mathbf{y} \times (\dot{\boldsymbol{\omega}}_p \times \mathbf{r}_2) - m_p \mathbf{x} \times \ddot{\mathbf{r}}_o = \mathbf{J}_p \cdot \dot{\boldsymbol{\omega}}_p - m_p \mathbf{x} \times (\dot{\boldsymbol{\omega}}_p \times \mathbf{r}_2) - m_p \mathbf{x} \times \ddot{\mathbf{r}}_o \\ = \mathbf{Q}^T \cdot \dot{\boldsymbol{\omega}}_p - m_p \mathbf{x} \times \ddot{\mathbf{r}}_o \end{aligned} \quad (4.33)$$

where

$$Q^T = I_p + m_p \tilde{y}\tilde{r}_2 = J_p - m_p \tilde{r}_2 \tilde{r}_2 + m_p \tilde{y}\tilde{r}_2 = J_p + m_p \tilde{x}\tilde{r}_2 \; . \quad (4.34)$$

Expansion of the elements of the matrix Q^T is given as the rate derivative coefficients of the detailed expansion in (4.28) above.

Letting

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T [\omega_o \cos\lambda_p t,\ \omega_o \sin\lambda_p t,\ \omega_3]^T$$

$$\dot{\boldsymbol{\omega}}_p = \mathbf{e}_p^T \lambda_p \omega_o [-\sin\lambda_p t,\ \cos\lambda_p t,\ 0]^T \; ,$$

where ω_o is sufficiently small that second-order terms may be neglected, the moments are:

$$\begin{aligned} M_1 = & \,\omega_o[\omega_3\{I_{33}^p - I_{22}^p + m_p(y_3 r_3 - y_2 r_2)\} - \lambda_p\{I_{11}^p - m_p(y_2 r_2 + y_3 r_3)\}]\sin\lambda_p t \\ & + \omega_o[(\omega_3 - \lambda_p)\{I_{12}^p - m_p y_2 r_1\}]\cos\lambda_p t + [I_{23}^p - m_p y_3 r_2]\omega_3^2 \end{aligned} \quad (4.35a)$$

$$\begin{aligned} M_2 = & \,\omega_o[\omega_3\{I_{11}^p - I_{33}^p + m_p(y_1 r_1 - y_3 r_3)\} + \lambda_p\{I_{22}^p - m_p(y_1 r_1 + y_3 r_3)\}]\cos\lambda_p t \\ & + \omega_o[(\lambda_p - \omega_3)\{I_{12}^p - m_p y_1 r_2\}]\sin\lambda_p t - [I_{13}^p - m_p y_3 r_1]\omega_3^2 \end{aligned} \quad (4.35b)$$

$$\begin{aligned} M_3 = & \,\omega_o[(\omega_3 - \lambda_p)\{I_{13}^p - m_p y_1 r_3\}\sin\lambda_p t - (\omega_3 + \lambda_p)\{I_{23}^p - m_p y_2 r_3\}\cos\lambda_p t] \\ & + [m_p(y_1 r_2 - y_2 r_1)]\omega_3^2 \; . \end{aligned} \quad (4.35c)$$

For the simple case where the platform is despun, $\omega_3 = 0$, both platform and rotor are statically balanced, and $I_{12}^p = 0$,

$$M_1 = -\omega_o\lambda_p[I_{11}^p - m_p y_3 r_3]\sin\lambda_p t = -\omega_o\lambda_p Q_{11}\sin\lambda_p t \quad (4.36a)$$

$$M_2 = \omega_o\lambda_p[I_{22}^p - m_p y_3 r_3]\cos\lambda_p t = \omega_o\lambda_p Q_{22}\cos\lambda_p t \quad (4.36b)$$

$$M_3 = -\omega_o\lambda_p[I_{13}^p \sin\lambda_p t + I_{23}^p \cos\lambda_p t] = \omega_o\lambda_p[Q_{13}\sin\lambda_p t + Q_{23}\cos\lambda_p t] \; . \quad (4.36c)$$

Lastly, if the platform is symmetric, $I_{11}^p \approx I_{22}^p \approx \sqrt{I_{11}^p I_{22}^p} = I_T^p$, the net moment is constant and equal to

$$M = \sqrt{M_1^2 + M_2^2} = \omega_o\lambda_p[I_T^p - m_p y_3 r_3] = \theta_n\lambda_p^2[I_T^p - m_p y_3 r_3] = \theta_n\lambda_p^2 Q_{11} \; . \quad (4.37)$$

Note when external forces are applied to the vehicle the $\mathbf{x} \times \ddot{\mathbf{r}}_o$ term from (4.24) must also be added to the moment. Also note that the equations derived can be applied to determine the restraint moments to hold any appendage with mass center at $\mathbf{r}_2$ and attach point $\mathbf{y}$.

4.3 Linearized Motion Induced by Combined Rotor Static and Dynamic Imbalance

The purpose of this derivation is to describe the trajectory of a point on the despun platform of a dual-spin spacecraft in the presence of combined rotor static and dynamic imbalance. Further, we wish to determine the torques acting along the gimbal axis of a gimbaled appendage on the despun platform. Extracting first-order dynamic imbalance torque terms from (2.22) and static imbalance torque terms from (3.42) expressed in the despun platform basis $\mathbf{e}_p$

$$\mathbf{B} = \mathbf{e}_p^T\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \mathbf{e}_p^T\begin{bmatrix} -\omega_s^2[J_{23}^s \cos\psi + J_{13}^s \sin\psi] \\ +\omega_s^2[J_{13}^s \cos\psi - J_{23}^s \sin\psi] \\ 0 \end{bmatrix} + \mathbf{e}_p^T m_s\omega_s^2\begin{bmatrix} y_e z_s \\ -x_e z_s \\ -(m_p/m)[x_e y_p - y_e x_p] \end{bmatrix} \tag{4.38}$$

$$= \mathbf{e}_p^T\begin{bmatrix} -\omega_s^2[J_{23}^s \cos\psi + J_{13}^s \sin\psi] \\ +\omega_s^2[J_{13}^s \cos\psi - J_{23}^s \sin\psi] \\ 0 \end{bmatrix} + \mathbf{e}_p^T m_s\omega_s^2\begin{bmatrix} z_s(y_s \cos\psi + x_s \sin\psi) \\ -z_s(x_s \cos\psi - y_s \sin\psi) \\ -(m_p/m)[(y_p x_s - x_p y_s)\cos\psi - (y_p y_s + x_p x_s)\sin\psi] \end{bmatrix}$$

$$= \mathbf{e}_p^T\omega_s^2\begin{bmatrix} -[J_{23}^s - z_s y_s m_s]\cos\psi - [J_{13}^s - z_s x_s m_s]\sin\psi \\ +[J_{13}^s - z_s x_s m_s]\cos\psi - [J_{23}^s - z_s y_s m_s]\sin\psi \\ -m_s(m_p/m)r_f r_e[\sin(\phi_p - \phi_s)\cos\psi - \cos(\phi_p - \phi_s)\sin\psi] \end{bmatrix}$$

$$= \mathbf{e}_p^T\omega_s^2\begin{bmatrix} -I_{23}^s \cos\psi - I_{13}^s \sin\psi \\ +I_{13}^s \cos\psi - I_{23}^s \sin\psi \\ \bar{I}_{12}^s \cos\psi + \bar{I}_{33}^s \sin\psi \end{bmatrix} = \mathbf{e}_p^T\omega_s^2\begin{bmatrix} -\sqrt{(I_{13}^s)^2 + (I_{23}^s)^2}\cos(\psi - \phi) \\ \sqrt{(I_{13}^s)^2 + (I_{23}^s)^2}\sin(\psi - \phi) \\ \sqrt{(\bar{I}_{12}^s)^2 + (\bar{I}_{33}^s)^2}\sin(\psi - \phi_s + \phi_p) \end{bmatrix}$$

where

$$\phi_d = \mathrm{Tan}^{-1}[I_{13}^s/I_{23}^s]\ ;\ \ \phi_s = \mathrm{Tan}^{-1}[y_s/x_s]\ ;\ \ \phi_p = \mathrm{Tan}^{-1}[y_p/x_p]\ . \tag{4.39}$$

It is evident from above that transverse torques on the platform can be nulled by cancellation of static and dynamic imbalances of the rotor by setting

$$z_s x_s m_s = J_{13}^s\ ;\ \ z_s y_s m_s = J_{23}^s \tag{4.40}$$

or alternatively stated, by nulling the rotor products of inertia seen from the spacecraft cm.

The approach here is to replace the imbalance with forces and torques on a balanced spacecraft which induce the same platform motion. We assert that the platform small displacement motion is the superposition of coning motion induced by the transverse imbalance torques and cylindrical translation induced by static imbalance. The torques are given by 4.38. Taking $I_{12} = \omega_p = 0$ and

$$\Delta_p = I_{11}I_{22}I_{33}^p(1 - r) \tag{4.41}$$

and expanding from (2.40, 44, and 45), the platform angular rates are

$$\omega_{p1}(t) = \omega_s\left[\frac{[I_{22}I_{33}^p - I_{23}^2]}{\Delta_p(1 - \sigma^2)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] + \omega_s\left[\frac{[\sigma_1 I_{11} I_{33}^p]}{\Delta_p(1 - \sigma^2)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \tag{4.42a}$$

$$+\ \omega_s\left[\frac{[\sigma_1 I_{23} I_{11}^p]}{\Delta_p(1 - \sigma^2)}\right][\bar{I}_{12}^s \cos\psi + \bar{I}_{33}^s \sin\psi]$$

$$\omega_{p2}(t) = \omega_s\left[\frac{[I_{11}I_{33}^p - I_{13}^2]}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] + \omega_s\left[\frac{[\sigma_2 I_{22}I_{33}^p]}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] \tag{4.42b}$$

$$- \omega_s\left[\frac{[\sigma_2 I_{13}I_{22}^p]}{\Delta_p(1-\sigma^2)}\right][\bar{I}_{12}^s \cos\psi + \bar{I}_{33}^s \sin\psi]$$

$$\omega_{p3}(t) = \omega_s\left[\frac{I_{23}I_{11}(1+\sigma_1)}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] + \omega_s\left[\frac{I_{13}I_{22}(1+\sigma_2)}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \tag{4.42c}$$

$$- \omega_s\left[\frac{I_{11}I_{22}(1-\sigma^2)}{\Delta_p(1-\sigma^2)}\right][\bar{I}_{33}^s \cos\psi - \bar{I}_{12}^s \sin\psi]$$

$$\omega_{p1}(t) \approx \omega_s\left[\frac{\{[I_{22}I_{33}^p - I_{23}^2] + [\sigma_1 I_{11}I_{33}^p]\}}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \tag{4.43a}$$

$$\omega_{p2}(t) \approx \omega_s\left[\frac{\{[I_{11}I_{33}^p - I_{13}^2] + [\sigma_2 I_{22}I_{33}^p]\}}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] \tag{4.43b}$$

$$\omega_{p3}(t) \approx - \omega_s\left[\frac{I_{11}I_{22}(1-\sigma^2)}{\Delta_p(1-\sigma^2)}\right][\bar{I}_{33}^s \cos\psi - \bar{I}_{12}^s \sin\psi] \tag{4.43c}$$

$$\dot{\omega}_{p1}(t) = - \omega_s^2\left[\frac{\{[I_{22}I_{33}^p - I_{23}^2] + [\sigma_1 I_{11}I_{33}^p]\}}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] \tag{4.44a}$$

$$\approx - \omega_s^2\left[\frac{1}{I_{11}(1-\sigma)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] \approx - \omega_s^2[\theta_{w2} \sin\psi + \theta_{w1} \cos\psi]$$

$$\dot{\omega}_{p2}(t) = \omega_s^2\left[\frac{\{[I_{11}I_{33}^p - I_{13}^2] + [\sigma_2 I_{22}I_{33}^p]\}}{\Delta_p(1-\sigma^2)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \tag{4.44b}$$

$$\approx \omega_s^2\left[\frac{1}{I_{22}(1-\sigma)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \approx \omega_s^2[\theta_{w2} \cos\psi - \theta_{w1} \sin\psi] \ .$$

$$\dot{\omega}_{p3}(t) \approx \omega_s^2\left[\frac{I_{11}I_{22}(1-\sigma^2)}{\Delta_p(1-\sigma^2)}\right][\bar{I}_{33}^s \sin\psi + \bar{I}_{12}^s \cos\psi] \approx \omega_s^2\left[\frac{1}{I_{33}^p}\right][\bar{I}_{33}^s \sin\psi + \bar{I}_{12}^s \cos\psi] \ . \tag{4.44c}$$

The response of a platform mounted axial accelerometer is simply

$$a_3(t) = \dot{\omega}_{p1}(t)r_2 - \dot{\omega}_{p2}(t)r_1 \approx - \omega_s^2\{r_2[\theta_{w2} \sin\psi + \theta_{w1} \cos\psi] + r_1[\theta_{w2} \cos\psi - \theta_{w1} \sin\psi]\} \ , \tag{4.45}$$

and clearly measurement of magnitude and phase of this acceleration yields the composite imbalance quantities I_{13}^s, I_{23}^s and no information to separate static and dynamic imbalance.

On Figure 3.1b is a representation of the cylindrical coning motion where it is illustrated that in the absence of torques the platform cm, and every point on the platform, traces a circle of radius $(m_s/m)r_e = (1 - m_p/m)r_e$ while the rotor traces a circle of radius $(1 - m_s/m)r_e$. Hence, a spacecraft with statically balanced rotor requires a rotor fixed radial force applied at the vehicle cm of magnitude $m_s\omega_s^2 r_e$ to induce this cylindrical coning motion. Deducing from the sketch of Figure 3.1b, the platform cm position trajectory is

$$\mathbf{r} = - \mathbf{e}_p^T(m_s/m)[x_s \cos\psi - y_s \sin\psi,\ y_s \cos\psi + x_s \sin\psi,\ 0]^T \tag{4.46}$$

$$= - \mathbf{e}_p^T(m_s/m)r_e[\cos(\psi + \phi_s),\ \sin(\psi + \phi_s),\ 0]^T \ ; \ \ \phi_s = \mathrm{Tan}^{-1}[y_s/x_s] \ .$$

from which we deduce the equivalent force required on a spacecraft with statically balanced rotor as

$$\mathbf{f} = m\ddot{\mathbf{r}} = \mathbf{e}_p^T m_s \omega_s^2 r_e [\cos(\psi + \phi_s), \sin(\psi + \phi_s), 0]^T \ . \tag{4.47}$$

Note that this force depends only on the static imbalance components and does not influence the axial acceleration measurement described above. Again consider the torques about the gimbal axis of an appendage. The platform transverse angular rates and derivatives thereof may be applied in Eq. 4.32 to evaluate appendage support torques induced by our pseudo torque. Next the expression $(m_a/m)\mathbf{x}\times\mathbf{f}$ (see 4.25) yields the reaction torques due to the pseudo force $\mathbf{f}$. The the vector component sum of these torques along the gimbal axis yields the gimbal reaction torque that must be supplied to hold the appendage fixed. In symbolic form this is repeated from (4.28) as

$$\mathbf{M}_b = \mathbf{I}_a \cdot \dot{\omega}_a + \omega_a \times [\mathbf{I}_a \cdot \omega_a] - m_a \mathbf{y} \times \ddot{\mathbf{r}}_2 - (m_a/m)\mathbf{x}\times\mathbf{f} \ , \tag{4.48}$$

where we use the sub a to denote the appendage under study. All terms of $\mathbf{M}_b$ projected on the gimbal axis can not be computed from the composite imbalance terms measured with the accelerometer. Hence, measurement of the appendage gimbal reaction or more practically the gimbal relative angle excursions yields the total $\mathbf{M}_b$ from which the $\mathbf{x}\times\mathbf{f}$ term can be separated and used to isolate the static imbalance components introduced separately by the pseudo force.

Now note that small values of dynamic and/or static imbalance will result in small linear range motion. A good approximation of the small angle behavior of the appendage can be obtained by assuming it is fixed to the platform, calculating the internal dynamic disturbance torque, and then applying the torque transmission of any appropriate gimbal control loop. When we linearize (4.47) and assume $\omega_a = \omega_p$, the result is

$$\mathbf{M}_b = \mathbf{I}_a \cdot \dot{\omega}_p - m_a \mathbf{y}\times\ddot{\mathbf{r}}_2 - (m_a/m)\mathbf{x}\times\mathbf{f} = \mathbf{J}_a \cdot \dot{\omega}_p - m_a \mathbf{x}\times\ddot{\mathbf{r}}_2 - (m_a/m)\mathbf{x}\times\mathbf{f} = \mathbf{e}_p^T [M_1, M_2, M_3]^T \tag{4.49}$$

$$= \mathbf{e}_p^T \begin{bmatrix} [I_{11}^a - m_a(y_2r_2 + y_3r_3)]\dot{\omega}_1 - [I_{12}^a - m_a y_2 r_1]\dot{\omega}_2 - [I_{13}^a - m_a y_3 r_1]\dot{\omega}_3 + (m_a/m)x_3 f_2 \\ -[I_{12}^a - m_a y_1 r_2]\dot{\omega}_1 + [I_{22}^a - m_a(y_1r_1 + y_3r_3)]\dot{\omega}_2 - [I_{23}^a - m_a y_3 r_2]\dot{\omega}_3 - (m_a/m)x_3 f_1 \\ -[I_{13}^a - m_a y_1 r_3]\dot{\omega}_1 - [I_{23}^a - m_a y_2 r_3]\dot{\omega}_2 + [I_{33}^a - m_a(y_1r_1 + y_2r_2)]\dot{\omega}_3 - (m_a/m)[x_1f_2 - x_2f_1] \end{bmatrix}$$

$$= \mathbf{e}_p^T \begin{bmatrix} [J_{11}^a - m_a(x_2r_2 + x_3r_3)]\dot{\omega}_1 - [J_{12}^a - m_a x_2 r_1]\dot{\omega}_2 - [J_{13}^a - m_a x_3 r_1]\dot{\omega}_3 + (m_a/m)x_3 f_2 \\ -[J_{12}^a - m_a x_1 r_2]\dot{\omega}_1 + [J_{22}^a - m_a(x_1r_1 + x_3r_3)]\dot{\omega}_2 - [J_{23}^a - m_a x_3 r_2]\dot{\omega}_3 - (m_a/m)x_3 f_1 \\ -[J_{13}^a - m_a x_1 r_3]\dot{\omega}_1 - [J_{23}^a - m_a x_2 r_3]\dot{\omega}_2 + [J_{33}^a - m_a(x_1r_1 + x_2r_2)]\dot{\omega}_3 - (m_a/m)[x_1f_2 - x_2f_1] \end{bmatrix} .$$

Expanding in terms of the imbalance induced angular rate and force solutions

$$M_1 = -\omega_s^2 \left[\frac{J_{11}^a - m_a(x_2r_2 + x_3r_3)}{I_{11}(1-\sigma)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] - \omega_s^2 \left[\frac{J_{12}^a - m_a x_2 r_1}{I_{22}(1-\sigma)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \tag{4.50a}$$

$$- \omega_s^2 \left[\frac{J_{13}^a - m_a x_3 r_1}{I_{33}^p}\right][\bar{I}_{33}^s \sin\psi + \bar{I}_{12}^s \cos\psi] + (m_a/m)x_3 f_2$$

$$= -\omega_s^2 \sqrt{(I_{13}^s)^2 + (I_{23}^s)^2}\,[a_{11}\cos(\psi - \phi) + a_{12}\sin(\psi - \phi)]$$

$$- \omega_s^2 m_p (m_s/m) r_f r_e a_{13} \sin(\psi - \phi_s + \phi_p) + \omega_s^2 (m_s m_a/m) x_3 r_e \sin(\psi + \phi_s)$$

$$= -\omega_s^2 \sqrt{(I_{13}^s)^2 + (I_{23}^s)^2}\sqrt{a_{11}^2 + a_{12}^2}\cos(\psi - \phi - \phi_1)$$

$$- \omega_s^2 (m_s r_e z_s)(m_p/m)(r_f/z_s) a_{13} \sin(\psi - \phi_s + \phi_p) + \omega_s^2 (m_s r_e z_s)(m_a/m)(x_3/z_s)\sin(\psi + \phi_s)$$

$$= -\omega_s^2 \sqrt{(J_{13}^s)^2 + (J_{23}^s)^2}\sqrt{a_{11}^2 + a_{12}^2}\cos(\psi - \phi_d - \phi_1) + \omega_s^2 (m_s z_s r_e)\sqrt{a_{11}^2 + a_{12}^2}\sin(\psi + \phi_s + \phi_1)$$

$$- \omega_s^2 (m_s r_e z_s)(m_p/m)(r_f/z_s) a_{13} \sin(\psi - \phi_s + \phi_p) + \omega_s^2 (m_s r_e z_s)(m_a/m)(x_3/z_s)\sin(\psi + \phi_s)$$

$$= -\omega_s^2 \{J_{13}^s a_{11} - J_{23}^s a_{12} - m_s x_s z_s [a_{11} + (m_a/m)(x_3/z_s)] + m_s y_s z_s a_{12} + (m_s r_e z_s)(m_p/m)(r_f/z_s) a_{13} \cos(\phi_p - \phi_s)\} \sin\psi$$

$$- \omega_s^2 \{J_{13}^s a_{12} + J_{23}^s a_{11} - m_s x_s z_s a_{12} - m_s y_s z_s [a_{11} + (m_a/m)(x_3/z_s)] - (m_s r_e z_s)(m_p/m)(r_f/z_s) a_{13} \sin(\phi_p - \phi_s)\} \cos\psi$$

$$M_2 = \omega_s^2\left[\frac{J_{12}^a - m_a x_1 r_2}{I_{11}(1-\sigma)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] + \omega_s^2\left[\frac{J_{22}^a - m_a(x_1 r_1 + x_3 r_3)}{I_{22}(1-\sigma)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \tag{4.50b}$$

$$- \omega_s^2\left[\frac{J_{23}^a - m_a x_3 r_2}{I_{33}^p}\right][\bar{I}_{33}^s \sin\psi + \bar{I}_{12}^s \cos\psi] - (m_a/m)x_3 f_1$$

$$= \omega_s^2\sqrt{(I_{13}^s)^2 + (I_{23}^s)^2}[a_{21}\cos(\psi - \phi) + a_{22}\sin(\psi - \phi)]$$

$$- \omega_s^2 m_p(m_s/m)r_f r_e a_{23}\sin(\psi - \phi_s + \phi_p) + \omega_s^2(m_s m_a/m)x_3 r_e\cos(\psi + \phi_s)$$

$$= \omega_s^2\sqrt{(J_{13}^s)^2 + (J_{23}^s)^2}\sqrt{a_{21}^2 + a_{22}^2}\cos(\psi - \phi - \phi_2) + \omega_s^2(m_s z_s r_e)[a_{21}\sin(\psi + \phi_s) + a_{22}\cos(\psi + \phi_s)]$$

$$- \omega_s^2 m_p(m_s/m)r_f r_e a_{23}\sin(\psi - \phi_s + \phi_p) + \omega_s^2(m_s m_a/m)x_3 r_e\cos(\psi + \phi_s)$$

$$M_3 = \omega_s^2\left[\frac{J_{13}^a - m_a x_1 r_3}{I_{11}(1-\sigma)}\right][I_{13}^s \sin\psi + I_{23}^s \cos\psi] - \omega_s^2\left[\frac{J_{23}^a - m_a x_2 r_3}{I_{22}(1-\sigma)}\right][I_{13}^s \cos\psi - I_{23}^s \sin\psi] \tag{4.50c}$$

$$- \omega_s^2\left[\frac{J_{33}^a - m_a(x_1 r_1 + x_2 r_2)}{I_{33}^p}\right][\bar{I}_{33}^s \sin\psi + \bar{I}_{12}^s \cos\psi] - (m_a/m)[x_1 f_2 - x_2 f_1]$$

$$= \omega_s^2\sqrt{(I_{13}^s)^2 + (I_{23}^s)^2}[a_{31}\cos(\psi - \phi) - a_{32}\sin(\psi - \phi)]$$

$$- \omega_s^2 m_p(m_s/m)r_f r_e a_{33}\sin(\psi - \phi_s + \phi_p) + \omega_s^2(m_s m_a/m)r_e\sqrt{x_1^2 + x_2^2}\cos(\psi + \phi_s - \lambda_3)$$

$$= \omega_s^2\sqrt{(J_{13}^s)^2 + (J_{23}^s)^2}\sqrt{a_{31}^2 + a_{32}^2}\cos(\psi - \phi - \phi_3) - \omega_s^2 m_s z_s r_e\sqrt{a_{31}^2 + a_{32}^2}\sin(\psi + \phi_s + \phi_3)$$

$$- \omega_s^2 m_p(m_s/m)r_f r_e a_{33}\sin(\psi - \phi_s + \phi_p) + \omega_s^2(m_s m_a/m)r_e\sqrt{x_1^2 + x_2^2}\cos(\psi + \phi_s - \lambda_3)$$

where we have defined the geometric parameter phase angles as

$$\lambda_3 = -\,\mathrm{Tan}^{-1}[x_1/x_2]\,;\;\; \phi_1 = \mathrm{Tan}^{-1}[a_{12}/a_{11}]\,;\;\; \phi_2 = \mathrm{Tan}^{-1}[a_{22}/a_{21}]\,;\;\; \phi_3 = -\,\mathrm{Tan}^{-1}[a_{32}/a_{31}]\,. \tag{4.50d}$$

We have gone somewhat to extremes in manipulating the M_i in different forms in order to show explicitly terms in total spacecraft dynamic imbalance I_{ij}^s due to rotor static and dynamic contributions, rotor dynamic imbalance J_{ij}^s, and static imbalance terms r_e and r_f respectively for rotor and platform static imbalance. We have already observed in (4.45) that only the composite dynamic imbalance with respect to the vehicle cm, I_{ij} is identifyible in the platform axial acceleration. In the second form of M_1 we may observe the potential separation of static and dynamic terms. Note the importance of the vehicle cm to rotor cm axial offset parameter z_s. As z_s increases the vehicle dynamic imbalance contributed by rotor static imbalance $m_s r_e z_s$ increases, while the translation term due to f_2 and the, perhaps already second-order, platform static imbalance term $m_p r_e r_f$ remain fixed. Similar remarks apply to the companion torques M_2, M_3. Hence, for sufficiently large z_s the latter terms become negligible in the gimbal torque and the rotor static and dynamic imbalance are inseparable by measuring axial acceleration and gimbal torque or pointing. Next consider the case of no rotor dynamic imbalance. Then the ratio of torque due to offset terms to torque induced by dynamic imbalance at the vehicle cm is

$$\frac{\omega_s^2(m_s/m)r_e\sqrt{(m_p r_f a_{13})^2 + (m_a x_3)^2}}{\omega_s^2(m_s z_s r_e)\sqrt{a_{11}^2 + a_{12}^2}} = \frac{(1/m)\sqrt{(m_p r_f a_{13})^2 + (m_a x_3)^2}}{z_s\sqrt{a_{11}^2 + a_{12}^2}}\,.$$

The denominator term is indistinguishable from dynamic imbalance and in practical cases will tend to dominate.

We next wish to investigate the behavior of pointing error induced in a gimbaled payload by the imbalance components. The roll M_1 component of error should be representative. Consider a simplified case where we consider only the the imbalance in a single plane, the 1-3 plane. Then the last form of M_1 in (4.50a) reduces to

$$M_1 = -\,\omega_s^2\{J_{13}^s a_{11} - m_s x_s z_s[a_{11} + (m_a/m)(x_3/z_s) - (m_p/m)(x_p/z_s)a_{13}]\}\sin\psi \tag{4.51}$$

$$-\,\omega_s^2\{J_{13}^s a_{12} - m_s x_s z_s[a_{12} + (m_p/m)(y_p/z_s)a_{13}]\}\cos\psi$$

$$= -\,\omega_s^2\{J_{13}^s a_{11} - \hat{J}_{13}^s[a_{11} + (m_a/m)(x_3/z_s) - (m_p/m)(x_p/z_s)a_{13}]\}\sin\psi$$

$$-\,\omega_s^2\{J_{13}^s a_{12} - \hat{J}_{13}^s[a_{12} + (m_p/m)(y_p/z_s)a_{13}]\}\cos\psi$$

$$= -\,\omega_s^2\{J_{13}^s a_{11} - \hat{J}_{13}^s[a_{11} + b_1]\}\sin\psi - \omega_s^2\{J_{13}^s a_{12} - \hat{J}_{13}^s[a_{12} + b_2]\}\cos\psi$$

If the magnitude of the sinusoidal gimbal torque $|M_1|$ is held constant the pointing disturbance will be constant. Forming this with spin speed normalized to unity

$$M_1^2 = \{J_{13}^s a_{11} - \hat{J}_{13}^s(a_{11} + b_1)\}^2 + \{J_{13}^s a_{12} - \hat{J}_{13}^s(a_{12} + b_2)\}^2 \tag{4.52}$$

Substitute $x = \hat{J}_{13}^s$ and $y = J_{13}^s$, such that

$$[(a_{11} + b_1)^2 + (a_{12} + b_2)^2]x^2 - 2[a_{11}(a_{11} + b_1) + a_{12}(a_{12} + b_2)]xy + [a_{11}^2 + a_{12}^2]y^2 = M_1^2\ . \tag{4.53}$$

This is the equation of a skewed ellipse centered at the origin of the x,y plane. Specifying a value for $|M_1|$ is equivalent to specifying a bound for sinusoidal pointing error induced by the 1-3 components of static and dynamic balance. The pointing error bound will be satisfied when the balance parameters are within the ellipse. Recall that the pointing error is a spin frequency sinusoid. Then balance errors in the orthogonal 2-3 plane produce pointing errors that are 90^o in time phase from the 1-3 plane contribution.

Consider an ellipse with major and minor axes aligned with a u,v coordinate basis described as

$$\frac{(u - u_o)^2}{a^2} + \frac{(v - v_o)^2}{b^2} = 1\ .$$

Rotating to a skewed basis

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x\cos\phi - y\sin\phi \\ y\cos\phi + x\sin\phi \end{bmatrix}$$

$$\frac{(x\cos\phi - y\sin\phi - u_o)^2}{a^2} + \frac{(y\cos\phi + x\sin\phi - v_o)^2}{b^2} = 1\ ,$$

and manipulating, we get the new form

$$\frac{x^2\cos^2\phi - 2xy\sin\phi\cos\phi + y^2\sin^2\phi - 2u_o(x\cos\phi - y\sin\phi) + u_o^2}{a^2}$$

$$+\ \frac{y^2\cos^2\phi + 2xy\sin\phi\cos\phi + x^2\sin^2\phi - 2v_o(y\cos\phi + x\sin\phi) + v_o^2}{b^2}$$

$$= [(\cos\phi/a)^2 + (\sin\phi/b)^2]x^2 + [(\sin\phi/a)^2 + (\cos\phi/b)^2]y^2 + 2\sin\phi\cos\phi[(1/b^2) - (1/a^2)]xy$$

$$-\ 2[(v_o/b^2)\sin\phi + (u_o/a^2)\cos\phi]x - 2[(v_o/b^2)\cos\phi - (u_o/a^2)\sin\phi]y + (u_o/a)^2 + (v_o/b)^2$$

$$= e_1x^2 + e_2y^2 + e_3xy + e_4x + e_5y + e_6 = 1\ ,$$

where $e_3^2 - 4e_1e_2 < 0$ is required for an ellipse. Then equating coefficients

$$e_1 = [(\cos\phi/a)^2 + (\sin\phi/b)^2] = [(a_{11} + b_1)^2 + [(a_{12} + b_2)^2]/M_1^2 \tag{4.54a}$$

$$e_2 = [(\sin\phi/a)^2 + (\cos\phi/b)^2] = [a_{11}^2 + a_{12}^2]/M_1^2 \tag{4.54b}$$

$$e_3 = 2\sin\phi\cos\phi[(1/b^2) - (1/a^2)] = -\,2[a_{11}(a_{11} + b_1) + a_{12}(a_{12} + b_2)]/M_1^2 \tag{4.54c}$$

$$e_4 = -\,2[(v_o/b^2)\sin\phi + (u_o/a^2)\cos\phi] = 0 \tag{4.54d}$$

$$e_5 = -2[(v_o/b^2)\cos\phi - (u_o/a^2)\sin\phi] = 0 \tag{4.54e}$$

$$e_6 = (u_o/a)^2 + (v_o/b)^2 = 0\ . \tag{4.54f}$$

Simultaneous solution of these nonlinear equations will yield the pointing error ellipse parameters. Excluding numerical solutions, this is a formidable set of equations to solve, but after a thwarted week of no-wind windsurfing on the North Shore of Maui we have happened upon the following solution. Note that the first three equations independently yield a, b, and ϕ. Also for out particular case, the last three require $u_o = v_o = 0$. Solving the first two for $1/a^2$ and $1/b^2$ respectively

$$\frac{1}{a^2} = \frac{e_1}{\cos^2\phi} - \frac{\tan^2\phi}{b^2} = \left[\frac{\cos^2\phi}{\cos^2\phi - \sin^2\phi}\right][e_1 - e_2\tan^2\phi] = \frac{e_1 - e_2\tan^2\phi}{1 - \tan^2\phi} \tag{4.55a}$$

$$\frac{1}{b^2} = \frac{e_2}{\cos^2\phi} - \frac{\tan^2\phi}{a^2} = \left[\frac{\cos^2\phi}{\cos^2\phi - \sin^2\phi}\right][e_2 - e_1\tan^2\phi] = \frac{e_2 - e_1\tan^2\phi}{1 - \tan^2\phi} \tag{4.55b}$$

$$\tan 2\phi = \frac{\sin 2\phi}{\cos 2\phi} = e_3/[e_2 - e_1] = \left[\frac{e_3}{[e_3^2 + (e_2 - e_1)^2]^{1/2}}\right]\left[\frac{[e_3^2 + (e_2 - e_1)^2]^{1/2}}{e_2 - e_1}\right]. \tag{4.55c}$$

Application of appropriate trigonometric identities gives

$$\tan^2\phi = \frac{[e_3^2 + (e_2 - e_1)^2]^{1/2} - (e_2 - e_1)}{[e_3^2 + (e_2 - e_1)^2]^{1/2} + (e_2 - e_1)}\ ;\ \ 1 - \tan^2\phi = \frac{2(e_2 - e_1)}{[e_3^2 + (e_2 - e_1)^2]^{1/2} + (e_2 - e_1)} \tag{4.56}$$

which may be used to complete the closed-form solution for ellipse axes a and b. Summarizing,

$$a^2 = 2/\{(e_2 + e_1) - [e_3^2 + (e_2 - e_1)^2]^{1/2}\} \tag{4.57a}$$

$$b^2 = 2/\{(e_2 + e_1) + [e_3^2 + (e_2 - e_1)^2]^{1/2}\} \tag{4.57b}$$

$$\phi = (1/2)\mathrm{Tan}^{-1}\{e_3/[e_2 - e_1]\}\ ;\ \ e_3^2 - 4e_1e_2 < 0 \tag{4.57c}$$

$$u_o = +a^2[e_5\sin\phi - e_4\cos\phi]/2 \tag{4.57d}$$

$$v_o = -b^2[e_5\cos\phi + e_4\sin\phi]/2\ ;\ \ e_6 = [e_4^2 + e_5^2]/4\ . \tag{4.57e}$$

Since the appendage pointing variation is fixed at spin frequency, the appendage pointing control loop transmission can be used to scale the required pointing bound to an admissible torque $|M_1|$. This combined with vehicle and appendage mass properties provides the ellipse on the plane of static versus dynamic imbalance in the rotor 1-3 plane. Imbalance in the 2-3 rotor coordinate plane will produce a disturbance torque that is similarly a spin frequency sinusoid that will be time phased 90^o from the 1-3 plane effect. Being sinusoids, the two pointing error components will combine in root sum square fashion.

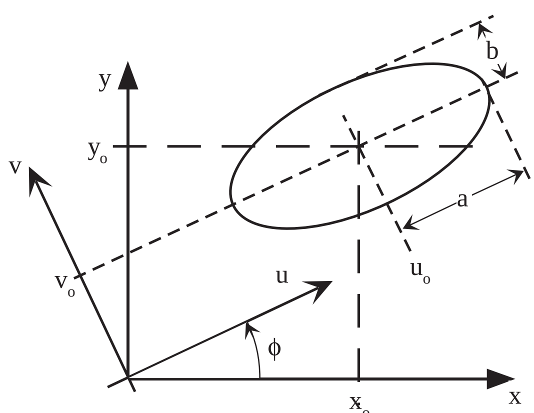
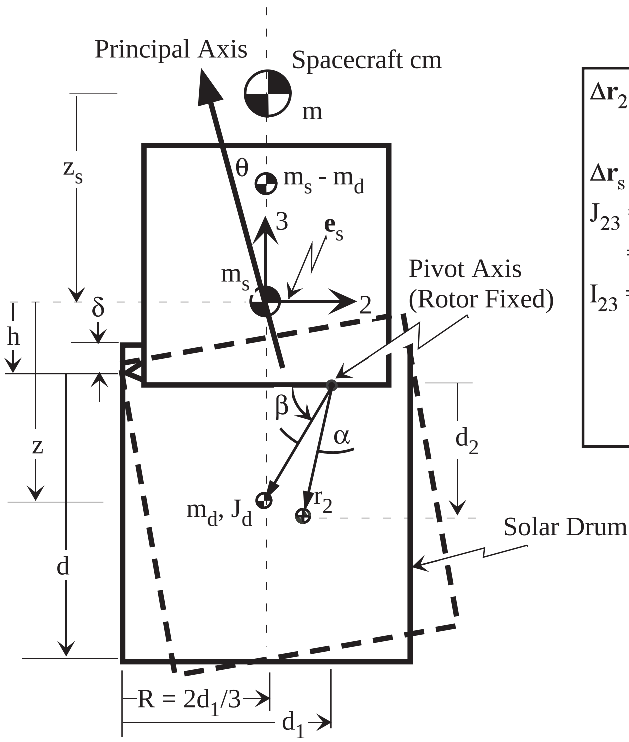


Figure 4.2 Definition of Ellipse Geometry.

In the analysis we have derived the roll torque M_1 that would be applied to a roll gimbal or payload elevation control. For a more general case it is a straightforward application of the above to get all three components of torque and the projection of this vector on the pointing gimbal axis. We alert the reader also that the derivation of torques assumes the gimbal angle is fixed which will result in an approximate solution, but does not detract from the insight provided by the solution.

On Hughes dual-spin spacecraft the rotor mounted telescoping solar drum is frequently used for rotor balancing. This is depicted on Figure 4.3. The telescoping drum is attached to the rotor by three rack and pinion deployment mechanisms at 120^o intervals on the periphery of the rotor. The rotor balance is adjusted by differentially extending one or two deployment mechanisms to tilt the drum as shown. On the figure we have defined geometric parameters and listed the sensitivity of static balance $\Delta\mathbf{r}_2$ and dynamic balance J_{23} and I_{23} with respect to rotor and vehicle cm respectively.



$$\Delta\mathbf{r}_2 = \mathbf{e}_s^T|\Delta r_2|[0, \sin\beta, -\cos\beta]^T = \mathbf{e}_s^T 2|r_2|\sin\alpha/2[0, \sin\beta, -\cos\beta]^T$$
$$= \mathbf{e}_s^T 2\sin\alpha/2[0, d_2, -d_1/3]^T \approx \mathbf{e}_s^T\alpha[0, d_2, -d_1/3]^T$$
$$\Delta\mathbf{r}_s = (m_d/m_s)\Delta\mathbf{r}_2;\ \Delta\mathbf{r}_{cm} = (m_s/m)\Delta\mathbf{r}_s = (m_d/m)\Delta\mathbf{r}_2$$
$$J_{23} = (z + \Delta z_d - \Delta z_s)(\Delta y_d - \Delta y_s)m_d + \Delta J_d \approx z(\Delta y_d - \Delta y_s)m_d + \Delta J_d$$
$$= \alpha z_s(1 - m_d/m_s)d_2 m_d + \alpha[J_{33} - J_{22}]$$
$$I_{23} = (z_s + \Delta z_s - \Delta z_{cm})(\Delta y_s - \Delta y_{cm})m_s + J_{23} \approx z_s(\Delta y_s - \Delta y_{cm})m_s + J_{23}$$
$$= z_s(1 - m_s/m)\Delta y_d m_d + J_{23} = \alpha z_s(1 - m_s/m)d_2 m_d + J_{23}$$
$$= \alpha[z_s(1 - m_s/m)d_2 m_d + z(1 - m_d/m_s)d_2 m_d + (J_{33} - J_{22})]$$
$$= [z_s(1 - m_s/m)d_2 m_d + z(1 - m_d/m_s)d_2 m_d + (J_{33} - J_{22})]\delta/d_1$$

Figure 4.3 Balance Concept Geometry and Sensitivity for Tilting Solar Drum on Dual-Spin Spacecraft.

5.0 Selected Solutions of the Dual-Spin Spacecraft Equations

5.1 Steady State Response to Rotor Dynamic Imbalance

5.1.1 Dual-Spin

Let the platform and rotor rates spin be $\omega_{p3} = \omega_p$, ω_s with corresponding rotor to platform relative rate spin $\omega_r = \omega_s - \omega_p$. Then assuming a symmetric rotor, $\Delta I_s = I^s_{22} - I^s_{11} = 0$, and $I^p_{12} = 0$, the linearized transverse axis torque equations (Eqs. 2.22a and b) in a platform fixed basis become: (platform also balanced)

$$I_{11}\dot{\omega}_{p1} + I_{11}\lambda_1\omega_{p2} = -[I^s_{23}\cos\omega_r t + I^s_{13}\sin\omega_r t]\omega_s^2 \tag{5.1a}$$

$$I_{22}\dot{\omega}_{p2} - I_{22}\lambda_2\omega_{p1} = [I^s_{13}\cos\omega_r t - I^s_{23}\sin\omega_r t]\omega_s^2 \ . \tag{5.1b}$$

Solving for the steady state platform angular rates

$$\begin{bmatrix}\omega_{p1}\\ \omega_{p2}\end{bmatrix} = \frac{\omega_s^2}{I_{11}I_{22}(\omega_r^2 - \lambda_p^2)}\begin{bmatrix}[I_{22}\omega_r + I_{11}\lambda_1][I^s_{13}\cos\omega_r t - I^s_{23}\sin\omega_r t]\\ [I_{11}\omega_r + I_{22}\lambda_2][I^s_{23}\cos\omega_r t + I^s_{13}\sin\omega_r t]\end{bmatrix} \tag{5.2}$$

$$\to \frac{-\omega_s}{{I^s_{33}}^2 - I_{11}I_{22}}\begin{bmatrix}(I_{22} + I^s_{33})[I^s_{13}\cos\omega_s t - I^s_{23}\sin\omega_s t]\\ (I_{11} + I^s_{33})[I^s_{23}\cos\omega_s t + I^s_{13}\sin\omega_s t]\end{bmatrix}; \ \ \omega_p \to 0 \ .$$

Transforming to the rotor basis via Eq. 1.10 with $\psi = \omega_r t$, and defining

$$\Delta I_p = I^p_{22} - I^p_{11} = I_{22} - I_{11} \ , \tag{5.3}$$

yields the rotor rates as

$$\begin{bmatrix}\omega_{s1}\\ \omega_{s2}\end{bmatrix} = \frac{\omega_s^2/2}{I_{11}I_{22}(\omega_r^2 - \lambda_p^2)}\begin{bmatrix}I^s_{13}[(I_{22} + I_{11})\omega_r + I_{11}\lambda_1 + I_{22}\lambda_2] + \Delta I_p\omega_r[I^s_{13}\cos 2\omega_r t - I^s_{23}\sin 2\omega_r t]\\ I^s_{23}[(I_{22} + I_{11})\omega_r + I_{11}\lambda_1 + I_{22}\lambda_2] - \Delta I_p\omega_r[I^s_{23}\cos 2\omega_r t + I^s_{13}\sin 2\omega_r t]\end{bmatrix} \tag{5.4}$$

$$\to \frac{-\omega_s}{2[{I^s_{33}}^2 - I_{11}I_{22}]}\begin{bmatrix}I^s_{13}[I_{11} + I_{22} + 2I^s_{33}] + \Delta I_p I^s_{13}\cos 2\omega_s t - \Delta I_p I^s_{23}\sin 2\omega_s t\\ I^s_{23}[I_{11} + I_{22} + 2I^s_{33}] - \Delta I_p I^s_{23}\cos 2\omega_s t - \Delta I_p I^s_{13}\sin 2\omega_s t\end{bmatrix}; \ \ \omega_p \to 0 \ .$$

The first term is the familiar constant rotor rate due to dynamic imbalance. The second term is a twice spin frequency disturbance induced by platform asymmetry.

The 3-axis acceleration at a point

$$\mathbf{r} = \mathbf{e}_s^T[r_1, r_2, r_3]^T \tag{5.5}$$

on the rotor is given by the 3-axis scalar equation of (4.5) as

$$a_3 = r_2\dot{\omega}_{s1} - r_1\dot{\omega}_{s2} + (r_1\omega_{s1} + r_2\omega_{s2})\omega_s - r_3(\omega_{s1}^2 + \omega_{s2}^2) \ . \tag{5.6}$$

Substituting the above transverse rates in a_3 gives

$$a_3 = \frac{(r_1 I^s_{13} + r_2 I^s_{23})[(I_{11} + I_{22})\omega_r + I_{11}\lambda_1 + I_{22}\lambda_2]\omega_s^3}{2I_{11}I_{22}(\omega_r^2 - \lambda_p^2)} \tag{5.7}$$

$$- \frac{3\Delta I_p\omega_s^3\omega_r}{2I_{11}I_{22}(\omega_r^2 - \lambda_p^2)}[(r_2 I^s_{23} - r_1 I^s_{13})\cos 2\omega_r t + (r_2 I^s_{13} + r_1 I^s_{23})\sin 2\omega_r t]$$

$$\to \frac{-(r_1 I^s_{13} + r_2 I^s_{23})(I_{11} + I_{22} + 2I^s_{33})\omega_s^2}{2[{I^s_{33}}^2 - I_{11}I_{22}]} + \frac{3\Delta I_p\omega_s^2\sqrt{r_1^2 + r_2^2}\sqrt{{I^s_{13}}^2 + {I^s_{23}}^2}}{2[{I^s_{33}}^2 - I_{11}I_{22}]}\cos(2\omega_s t - \gamma) \ ; \ \ \omega_p \to 0 \ ,$$

with

$$\gamma = \mathrm{Tan}^{-1}[(r_2 I^s_{13} + r_1 I^s_{23})/(r_2 I^s_{23} - r_1 I^s_{13})] \ . \tag{5.8}$$

The magnitude of the constant component of transverse rate is

$$\theta_w \omega_s = \frac{\omega_s^2/2[(I_{22}+I_{11})\omega_r + I_{11}\lambda_1 + I_{22}\lambda_2]\sqrt{I_{13}^{s\,2} + I_{23}^{s\,2}}}{I_{11}I_{22}(\omega_r^2 - \lambda_p^2)} \rightarrow \frac{\omega_s(I_{11}+I_{22}+2I_{33}^s)\sqrt{I_{13}^{s\,2}+I_{23}^{s\,2}}}{2[I_{33}^{s\,2} - I_{11}I_{22}]} \tag{5.9}$$

where θ_w is the wobble angle or the displacement of the principal axis of inertia from the bearing axis. Approximating $I_{11} \approx I_{22} \approx I_T$, gives the rotor imbalance wobble angle as

$$\theta_w \approx \frac{\sqrt{I_{13}^{s\,2}+I_{23}^{s\,2}}}{I_T - I_{33}^s} = \frac{\sqrt{I_{13}^{s\,2}+I_{23}^{s\,2}}}{I_T(1-\sigma)} \,. \tag{5.10}$$

Then the acceleration may be written

$$a_3 \approx a_o + \left[\frac{3r_o\omega_s^2\Delta I_p}{2(I_{33}^s + I_T)}\right]\theta_w \cos(2\omega_s t - \gamma) \,. \tag{5.11}$$

Hence, the twice spin rate component is proportional to imbalance magnitude. The two components of imbalance can be determined by correlating the phase of this signal with some spin phase reference, e.g., a sun sensor. Note the dc component of acceleration

$$a_o \approx \frac{(r_1 I_{13}^s + r_2 I_{23}^s)\omega_s^2}{I_T - I_{33}^s} \tag{5.12}$$

senses only the component of imbalance in the plane containing the spin axis and the radial line to the accelerometer. The dc acceleration will also contain a first-order corrupting term due to instrument misalignment. Hence, measurement of a_o is usually not a feasible way to detect imbalance.

5.1.2 All Spun

For a single spinning body with $I_{12} = 0$, the linearized torque equations are:

$$I_{11}\dot{\omega}_{s1} + (I_{33} - I_{22})\omega_s\omega_{s2} = \ \ I_{23}\omega_s^2 \tag{5.13a}$$

$$I_{22}\dot{\omega}_{s2} - (I_{33} - I_{11})\omega_s\omega_{s1} = -I_{13}\omega_s^2 \,. \tag{5.13b}$$

These result in steady state transverse rates

$$\begin{bmatrix}\omega_{s1}\\ \omega_{s2}\end{bmatrix} = \omega_s\begin{bmatrix} I_{13}/(I_{11}-I_{33}) \\ I_{23}/(I_{22}-I_{33})\end{bmatrix} \tag{5.14}$$

and linearized acceleration

$$a_3 = [r_1 I_{13}/(I_{11}-I_{33}) + r_2 I_{23}/(I_{22}-I_{33})]\omega_s^2 \,, \tag{5.15}$$

and for $I_{11} \approx I_{22} \approx \sqrt{I_{11}I_{22}} = I_T$

$$a_3 \approx [r_1 I_{13} + r_2 I_{23}]\omega_s^2/(I_T - I_{33})] \,. \tag{5.16}$$

Computing the steady state angular momentum

$$H = I[\omega_{s1}, \omega_{s2}, \omega_s]^T = [H_1, H_2, H_3]^T \,, \tag{5.17}$$

the wobble or spin axis tilt is found as

$$\theta_w = \mathrm{Tan}^{-1}[\sqrt{H_1^2 + H_2^2}/H_3] \approx \mathrm{Tan}^{-1}[\sqrt{H_1^2+H_2^2}/I_{33}\omega_s]$$

$$= \mathrm{Tan}^{-1}\sqrt{\{I_{13}/(I_{11}-I_{33})\}^2 + \{I_{23}/(I_{22}-I_{33})\}^2} \,. \tag{5.18}$$

Approximating transverse inertia symmetry and small tilt angle, respectively

$$\theta_w = \mathrm{Tan}^{-1}[\sqrt{I_{13}^{\,2} + I_{23}^{\,2}}/(I_T - I_{33})] \approx \sqrt{I_{13}^{\,2}+I_{23}^{\,2}}/(I_T - I_{33})] \,. \tag{5.19}$$

5.1.3 Wobble on a Multiple Rotor Vehicle

Here we develop a simple approximation for wobble of a multiple rotor vehicle where a dynamic imbalance product of inertia exists on one rotor. The development draws on the multiple rotor equations of Section 3.3 and is somewhat intuitive; therefore, awaits a rigorous development or verification by simulation. Using Eq. 3.49 and the expansion of Eq. 2.22c we get angular rate terms only for the 1-axis torque equation as

$$-\sum I_i\omega_i\omega_2 + I_{22}\omega_2\omega_s - I_{23}\omega_s^2 = 0\ .$$

This equation is expressed in the unbalanced rotor having product I_{23} and spin rate ω_s. All remaining bodies are assumed balanced and symmetric, and in steady spin we assume angular accelerations vanish in the unbalance rotor. The constant transverse rate in this body due to dynamic unbalance is then

$$\omega_2 = \frac{\omega_s I_{23}}{I_{22} - \Sigma\omega_i/\omega_2} = \frac{\omega_s I_{23}}{I_T - \Sigma I_i\omega_i/\omega_s} = \frac{\omega_s I_{23}}{I_T - H/\omega_s} = \frac{\omega_s I_{23}}{I_T(1-\lambda_o/\omega_s)} = \theta_w\,\omega_s\ ,$$

where transverse inertia symmetry is approximated when I_T is introduced. As is now well-known for dual-spin vehicles, it is clearly desirable to avoid a body spin rate approaching inertial nutation frequency. Further note that having the transverse rate in the spinning unbalanced body, constant in that body, the transverse rate in any other rotor or platform will be sinusoidal with the same magnitude and at relative rate. Acceleration at a point is straightforward using the resultant transverse rate.

5.2 Closed-Loop Response to Rotor Dynamic Imbalance

Open-loop response to rotor dynamic imbalance (wobble) was calculated for the steady-state in the above. In this section a technique for approximating the closed-loop response to imbalance is developed for the steady state case.

The dominant rotor imbalance torque terms from Eq. 2.22a and b are:

$$\mathbf{e}_p^T\begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix} = \mathbf{e}_p^T\begin{bmatrix} -I_{23}^s\cos\omega_s t - I_{13}^s\sin\omega_s t \\ I_{13}^s\cos\omega_s t - I_{23}^s\sin\omega_s t \end{bmatrix}\omega_s^2 \tag{5.20}$$

$$= \mathbf{e}_p^T\omega_s^2\sqrt{{I_{13}^s}^2 + {I_{23}^s}^2}\begin{bmatrix} -\cos\{\omega_s t - \phi\} \\ \sin\{\omega_s t - \phi\} \end{bmatrix}$$

where

$$\phi = \mathrm{Tan}^{-1}\{I_{13}^s/I_{23}^s\}\ . \tag{5.21}$$

The phase is inconsequential to calculation of the steady state response, thus transforming:

$$\mathbf{e}_p^T\begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix} = \mathbf{e}_p^T\omega_s^2\sqrt{{I_{13}^s}^2 + {I_{23}^s}^2}\,\{-s/\omega_s, -1\}^T\{\omega_s/(s^2+\omega_s^2)\}\ . \tag{5.22}$$

To calculate vehicle response, these torques are approximated as external forcing torques. It is convenient to express the two constrained components of torque as a single input in terms of the average rigid body open-loop wobble angle.

$$\theta_w = \sqrt{{I_{13}^s}^2 + {I_{23}^s}^2}/\{\sqrt{I_{11}I_{22}} - I_{33}^s\} = \sqrt{{I_{13}^s}^2 + {I_{23}^s}^2}/\{I_T(1-\sigma)\}\ . \tag{5.23}$$

The resultant single input representation is shown as Figure 5.1. The plant matrix is given by Equation 2.40, where

$$\omega = P(s)T\ . \tag{5.24}$$

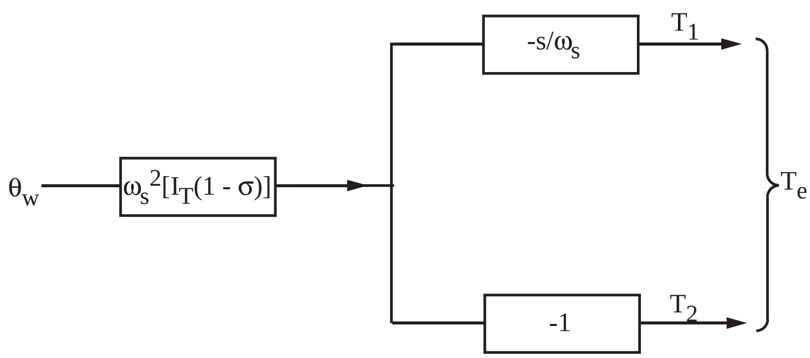


Figure 5.1 Rotor Imbalance Equivalent Torque Input Structure.

Letting M and F be respectively the measurement and feedback matrices, the system open-loop transmission is:

$$L = PG/s + PFM/s \, . \tag{5.25}$$

Substituting

$$T = -\,FM\theta + F\theta_c + T_e = -\,G\theta + F\theta_c + T_e \tag{5.26}$$

in ω, using $\theta = \omega/s$, and solving

$$\omega = \{I + L\}^{-1}P[F\theta_c + T_e] \tag{5.27}$$

in the system as diagrammed in Figure 5.2. Although perhaps tedious, the calculation of any variable response to imbalance torques is now straightforward. One approach used by this writer to avoid the extensive algebra in calculating ω above is to evaluate P, F, and G numerically and perform the manipulations and inversions numerically on a computer.

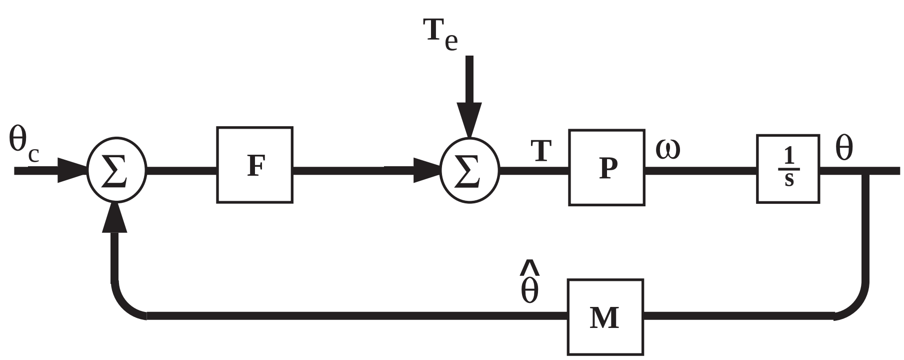


Figure 5.2 Closed-Loop Representation of Dual-Spin Vehicle with Control.

5.3 Equivalence of Bearing Misalignment and Rotor Imbalance

Position offset and/or angular misalignment of the despin bearing of a dual-spin vehicle is entirely indistinguishable from, or equivalent to imbalance. The geometry of bearing axis errors is depicted on Figure 5.3. Here we note that an offset δ results in static imbalance δm_s and angular misalignment of the bearing axis produces both static and dynamic imbalance components.

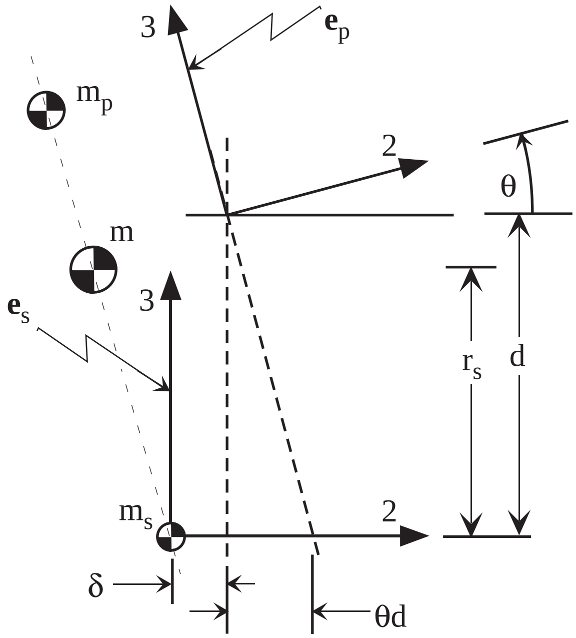


Figure 5.3 Imbalance Components Induced by Bearing Misalignment.

Employing the geometry of the figure, the static and dynamic imbalance components are respectively (arbitrarily assuming components are in the 2-3 plane)

$$S = (\delta - d\theta)m_s$$

$$\hat{J}^s_{23} = [J^s_{33} - J_{22}]\theta$$

Combining these components and translating to the vehicle cm, the equivalent dynamic imbalance about the vehicle cm becomes

$$\hat{I}^s_{23} = \delta r_s m_s + [J^s_{33} - J^s_{22} - d r_s m_s]\theta \ .$$

5.4 Coning Response to Constant Rate Rotation of an Imbalanced Platform

Assume a dynamically and statically balanced and symmetric rotor, a coordinate basis chosen so $I_{12} = 0$, and constant platform rotation rate $\omega_p(\omega_3 = 0)$. Eqs. 2.34a and b may be solved for steady state platform rates in $\mathbf{e}_p$, yielding

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T[\omega_1,\ \omega_2,\ \omega_3]^T = \mathbf{e}_p^T\omega_p[-\omega_p I_{13}/\lambda_2 I_{22},\ -\omega_p I_{23}/\lambda_1 I_{11},\ 1]^T \tag{5.28}$$

where λ_1, λ_2 are nutation frequency components given by (2.35a and b). The resultant vehicle angular momentum is

$$\mathbf{H} = \mathbf{e}_p^T[H_1,\ H_2,\ H_3]^T = \mathbf{e}_p^T\begin{bmatrix} -\omega_p I_{13}[\omega_p I_{11}/\lambda_2 I_{22} + 1] \\ -\omega_p I_{23}[\omega_p I_{22}/\lambda_1 I_{11} + 1] \\ I_{33}^p\omega_p + I_{33}^s\omega_s + \omega_p^2[I_{13}^2/\lambda_2 I_{22} + I_{23}^2/\lambda_1 I_{11}] \end{bmatrix} \tag{5.29}$$

and the cone angle is obtained as

$$\theta_c = \mathrm{Tan}^{-1}[\sqrt{H_1^2 + H_2^2}/H_3]\ . \tag{5.30}$$

Approximating the platform transverse inertias equal as $I_{11} \approx I_{22} \approx I_T = \sqrt{I_{11}I_{22}}$; $\lambda_1 = \lambda_2 = \lambda$, and the momentum simplifies to

$$\mathbf{H} = \mathbf{e}_p^T\begin{bmatrix} -\omega_p I_{13}[\omega_p/\lambda + 1] \\ -\omega_p I_{23}[\omega_p/\lambda + 1] \\ I_{33}^p\omega_p + I_{33}^s\omega_s + (\omega_p^2/\lambda I_T)[I_{13}^2 + I_{23}^2] \end{bmatrix} \tag{5.31}$$

while the cone angle becomes closely

$$\theta_c \approx \mathrm{Tan}^{-1}[(\omega_p/\lambda + 1)\omega_p\sqrt{I_{13}^2 + I_{23}^2}\,]/[I_{33}^p\omega_p + I_{33}^s\omega_s] \tag{5.32}$$

$$= \mathrm{Tan}^{-1}\{[\omega_p\sqrt{I_{13}^2 + I_{23}^2}\ ]/\lambda I_T\} = \mathrm{Tan}^{-1}[\sqrt{\omega_1^2 + \omega_2^2}/\omega_p]\ .$$

If the coning momentum is small, momentum is approximately conserved on the spin axis during spinup, i.e., $I_{33}^s\omega_{so} = I_{33}^s\omega_s + I_{33}^p\omega_p$. This results in $\lambda = (I_{33}^s/I_T)\omega_{so} - \omega_p = \sigma\omega_{so} - \omega_p$, and a reduction of the cone angle to

$$\theta_c = \mathrm{Tan}^{-1}\{[\sqrt{I_{13}^2 + I_{23}^2}/I_T][\omega_p/(\sigma\omega_{so} - \omega_p)]\}\ . \tag{5.33}$$

Transforming the platform rate to the rotor basis $\mathbf{e}_s$ and computing the 3-axis coning acceleration at a rotor fixed point $\mathbf{r} = \mathbf{e}_s^T[r_1,\ r_2,\ r_3]^T$ assuming ω_1, ω_2 constant gives (from Eq. 4.5)

$$a_1 = -\, r_3(\dot\psi - \omega_s)[\omega_1 \cos\psi + \omega_2 \sin\psi] - r_2\dot\omega_s - r_1\omega_s^2 \tag{5.34a}$$

$$-\ \omega_1(r_1\omega_1 + r_2\omega_2)\sin^2\psi - \omega_2(r_1\omega_2 - r_2\omega_1)\cos^2\psi - (r_2\omega_1^2 - r_2\omega_2^2 - 2r_1\omega_1\omega_2)\sin\psi\cos\psi$$

$$a_2 = r_3(\dot\psi - \omega_s)[\omega_2 \cos\psi - \omega_1 \sin\psi] + r_1\dot\omega_s - r_2\omega_s^2 \tag{5.34b}$$

$$-\ \omega_2(r_1\omega_1 + r_2\omega_2)\sin^2\psi + \omega_1(r_1\omega_2 - r_2\omega_1)\cos^2\psi - (r_1\omega_1^2 - r_1\omega_2^2 + 2r_2\omega_1\omega_2)\sin\psi\cos\psi$$

$$a_3 = (\dot\psi + \omega_s)[(r_1\omega_2 - r_2\omega_1)\sin\psi + (r_2\omega_2 + r_1\omega_1)\cos\psi] - r_3(\omega_1^2 + \omega_2^2). \tag{5.34c}$$

The constant and twice relative frequency terms are second-order in ω_i which is proportional to despun product of inertia over total transverse inertia. Hence, in most applications these can be neglected, and substituting for the angular rates, the accelerations reduce to

$$a_1 \approx r_3\omega_p^2(\dot\psi - \omega_s)[(I_{13}/\lambda_2 I_{22})\cos\psi + (I_{23}/\lambda_1 I_{11})\sin\psi] - r_2\dot\omega_s - r_1\omega_s^2 \tag{5.35a}$$

$$a_2 \approx r_3\omega_p^2(\dot\psi - \omega_s)[(I_{13}/\lambda_2 I_{22})\sin\psi - (I_{23}/\lambda_1 I_{11})\cos\psi] + r_1\dot\omega_s - r_2\omega_s^2 \tag{5.35b}$$

$$a_3 \approx \omega_p^2(\dot\psi + \omega_s)\{[r_2 I_{13}/\lambda_2 I_{22} - r_1 I_{23}/\lambda_1 I_{11}]\sin\psi - [r_1 I_{13}/\lambda_2 I_{22} + r_2 I_{23}/\lambda_1 I_{11}]\cos\psi\}. \tag{5.35c}$$

Since $\dot\psi = \omega_s - \omega_p$ the first two terms are approximately proportional to ω_p^3 while the last term is proportional to $\omega_p^2(2\omega_s - \omega_p)$. During platform superspin maneuvers it is sometimes convenient to set $\omega_s = \omega_p/2$ to allow the rotor mounted accelerometer to display nutation without corruption by platform coning. By a proper choice of vector basis the magnitude of relative rate sinusoidal axial acceleration $a_3(t)$ reduces to

$$a_c = (2\omega_s - \omega_p)\omega_p r_o \theta_c \ . \tag{5.36}$$

The above results are correct for a statically unbalanced platform provided the appropriate generalized inertia from Eq. 3.13 is employed, however; the acceleration expansion does not include the effect of the resultant vehicle cm offset due to platform static imbalance. Four additional acceleration terms arise due to this effect and these are identified in Eq. 4.5 and expanded in Eqs. 4.12 through 4.15. The additional spin axis(3-axis) term for imbalanced platform coning has been expanded and found to reduce quite simply to $\omega_p[\omega_1\delta_1 + \omega_2\delta_2]$, where ω_1, ω_2 are the platform coning rates and δ_1, δ_2, are the *vehicle* radial cm offset dimensions.

5.5 Free (Nutation) Response to a Transverse Torque Impulse with Asymmetric Unbalanced Platform

Consider the torque impulse response for a dual-spin vehicle with balanced and symmetric rotor, despun platform ($\omega_p = 0$), and platform basis $\mathbf{e}_p$ chosen such that $I_{12} = 0$. Then the dynamics as given by Eq. 2.40 apply and from (2.35), (2.47d and e) platform nutation frequency is

$$\lambda_p^2 = \lambda_1\lambda_2/(1 - r) = (I_s\omega_s)^2/\{I_{11}I_{22}(1 - r)\}. \tag{5.36}$$

We apply a torque impulse T_1 ft-lb-sec about the 1-axis (or equivalently the initial condition $T_1 = I_{11}\omega_1(0)$) and compute the response. From (2.40)

$$P_{11} = [I_{22}I_{33}^p - I_{23}^2]s^2/\Delta \tag{5.37a}$$

$$P_{21} = [I_{13}I_{23}][s + I_{22}I_{33}^p\lambda_2/I_{13}I_{23}]s/\Delta \tag{5.37b}$$

$$P_{31} = [I_{13}I_{22}][s + I_{23}\lambda_2/I_{13}]s/\Delta \tag{3.37c}$$

with Δ given by (2.47d). Applying the impulse and inverting the transform, the platform rates in $\mathbf{e}_p$ are

$$\omega_1 = T_1[\{I_{22}I_{33}^p - I_{23}^2\}/\Delta'] \cos\lambda_p t = a\cos\lambda_p t \tag{5.38a}$$

$$= \{(b + a)/2 - (b - a)/2\} \cos\lambda_p t$$

$$\omega_2 = T_1[\{I_{22}I_{33}^p\lambda_2/\lambda_p\}/\Delta'] \sin\lambda_p t + T_1[\{I_{13}I_{23}\}/\Delta'] \cos\lambda_p t = b\sin\lambda_p t + \alpha\cos\lambda_p t \tag{5.38b}$$

$$= \{(b + a)/2 + (b - a)/2\} \sin\lambda_p t + \alpha\cos\lambda_p t$$

$$\omega_3 = T_1[\{I_{22}I_{13}\}/\Delta'] \cos\lambda_p t + T_1[\{I_{23}I_{22}\lambda_2/\lambda_p\}/\Delta'] \sin\lambda_p t \tag{5.38c}$$

$$= c\cos\lambda_p t + d\sin\lambda_p t$$

where a, b, c, d, and α are now defined and

$$\Delta' = I_{11}I_{22}I_{33}^p(1 - r)\ . \tag{5.39}$$

For completeness and easy reference we also invert P_{12}, P_{22}, P_{32} to get the response to a torque inpulse T_2 as

$$\omega_1 = T_2[\{I_{13}I_{23}\}/\Delta'] \cos\lambda_p t - T_2[\{I_{11}I_{33}^p\lambda_1/\lambda_p\}/\Delta'] \sin\lambda_p t$$

$$\omega_2 = T_2[\{I_{11}I_{33}^p - I_{13}^2\}/\Delta'] \cos\lambda_p t$$

$$\omega_3 = T_2[\{I_{11}I_{23}\}/\Delta'] \cos\lambda_p t - T_2[\{I_{13}I_{11}\lambda_1/\lambda_p\}/\Delta'] \sin\lambda_p t\ .$$

Integrating the rates for T_1

$$\lambda_p\theta_1 = a\sin\lambda_p t \tag{5.40a}$$

$$\lambda_p\theta_2 = b(1 - \cos\lambda_p t) + \alpha\sin\lambda_p t\ . \tag{5.40b}$$

We consider the balanced case briefly. The motion of the platform spin axis is as shown by Figure 5.4a below. For this case r = 0, and the motion describes an ellipse given by

$$(\lambda_p\theta_1)^2/a + (\lambda_p\theta_2 - b)^2/b = 1\ , \tag{5.41}$$

with

$$b/a = \lambda_2/\lambda_p = \sqrt{I_{11}/I_{22}}\ . \tag{5.42}$$

If we applied the same torque impulse about the 2-axis, the resultant ellipse is symmetric about the 2-axis with major axis $T_2/\sqrt{I_{11}I_{22}}$ along the 2-axis and minor axis T_2/I_{22} parallel and above the 1-axis of Figure 5.4a as shown by the dashed path. Study of the drawing and the preceding discussion reveals that the nutation ellipse always has one axis of length $T/\sqrt{I_{11}I_{22}}$ along the momentum change vector, and this is the minor axis when torque is applied to the minimum inertia, and the major axis when torque is applied to the maximum inertia.

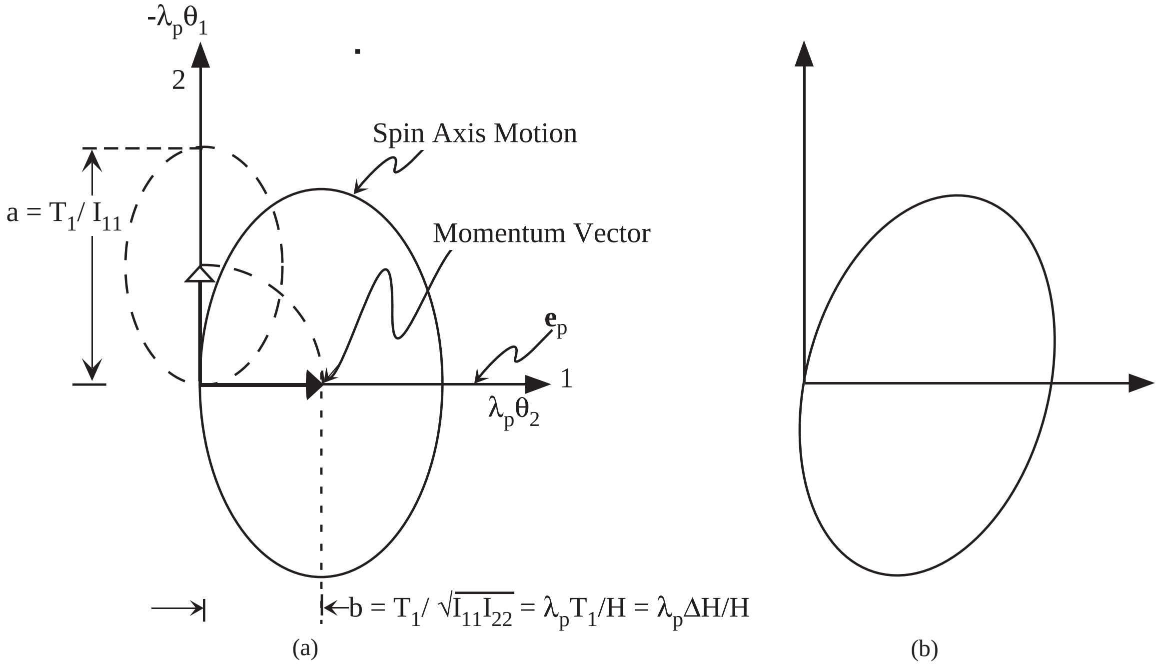


Figure 5.4 Spin Axis Motion Due to Impulse T_1 for $I_{22} > I_{11}$.

If the platform is unbalanced in the plane of the impulse only the same motion results. If it is unbalanced in both planes a more complex motion occurs which is an ellipse traced about a moving center and has the general appearance of an ellipse with axes rotated in the basis $\mathbf{e}_p$ as indicated by the example of Figure 5.4b.

Next, using (1.10) to transform the rate to the rotor basis $\mathbf{e}_s$, and denoting rotor nutation frequency as $\lambda_s = \lambda_p - \omega_s = (\sigma - 1)\omega_s$, the rotor rates are

$$\omega_{s1} = b\cos\lambda_s t - (b-a)\cos\lambda_p t\cos\omega_s t + \alpha\cos\lambda_p t\sin\omega_s t \tag{5.43a}$$

$$= \{(b+a)/2\}\cos\lambda_s t - (\alpha/2)\sin\lambda_s t - \{(b-a)/2\}\cos\{(\lambda_p+\omega_s)t\} + (\alpha/2)\sin\{(\lambda_p+\omega_s)t\}$$

$$= A\cos(\lambda_s t + \beta_1) - B\cos\{(\lambda_p+\omega_s)t + \beta_2\}$$

$$\omega_{s2} = a\sin\lambda_s t + (b-a)\sin\lambda_p t\cos\omega_s t + \alpha\cos\lambda_p t\cos\omega_s t \tag{5.43b}$$

$$= \{(b+a)/2\}\sin\lambda_s t + (\alpha/2)\cos\lambda_s t + \{(b-a)/2\}\sin\{(\lambda_p+\omega_s)t\} + (\alpha/2)\cos\{(\lambda_p+\omega_s)t\}$$

$$= A\sin(\lambda_s t + \beta_1) + B\sin\{(\lambda_p+\omega_s)t + \beta_2\}$$

where

$$2A = \sqrt{(b+a)^2 + \alpha^2} \tag{5.44a}$$

$$2B = \sqrt{(b-a)^2 + \alpha^2} \tag{5.44b}$$

$$\tan\beta_1 = \alpha/(b+a) \tag{5.44c}$$

$$\tan\beta_2 = \alpha/(b-a) \tag{5.44d}$$

$$b + a = T_1[I_{22}I_{33}^{p}(1 + \sqrt{I_{11}/I_{22}}) - I_{23}^{2}]/\Delta' \tag{5.44e}$$

Thus, while platform rate components are all at platform nutation frequency λ_p, the rotor rate contains components both at rotor nutation frequency $\lambda_s = \lambda_p - \omega_s = (\sigma - 1)\omega_s$, and at $\lambda_p + \omega_s = (\sigma + 1)\omega_s$.

Using Eq. 4.5, the resultant linear acceleration at a point

$$\mathbf{r}_a = \mathbf{e}_s^T[r_1, r_2, r_3]^T \tag{5.45}$$

is found as

$$a_1 = r_3\lambda_p[A\cos(\lambda_s t + \beta_1) + B\cos\{(\lambda_p + \omega_s)t + \beta_2\}] + r_2\lambda_p[c\sin\lambda_p t - d\cos\lambda_p t] - r_1\omega_s^2 \tag{5.46a}$$

$$= r_3\sigma\omega_s[A\cos\{(\sigma - 1)\omega_s t + \beta_1\} + B\cos\{(\sigma + 1)\omega_s t + \beta_2\}] + r_2\sigma\omega_s[c\sin\lambda_p t - d\cos\lambda_p t] - r_1\omega_s^2$$

$$a_2 = r_3\lambda_p[A\sin(\lambda_s t + \beta_1) - B\sin\{(\lambda_p + \omega_s)t + \beta_2\}] - r_1\lambda_p[c\sin\lambda_p t - d\cos\lambda_p t] - r_2\omega_s^2 \tag{5.46b}$$

$$= r_3\sigma\omega_s[A\sin\{(\sigma - 1)\omega_s t + \beta_1\} - B\sin\{(\lambda_p + \omega_s)t + \beta_2\}] - r_1\lambda_p[c\sin\lambda_p t - d\cos\lambda_p t] - r_2\omega_s^2$$

$$a_3 = A(\omega_s - \lambda_s)[r_1\cos(\lambda_s t + \beta_1) + r_2\sin(\lambda_s t + \beta_1)] - B(\lambda_p + 2\omega_s)[r_1\cos\{(\lambda_p + \omega_s)t + \beta_2\} \tag{5.46c}$$

$$+ r_2\sin\{(\lambda_p + \omega_s)t + \beta_2\}]$$

$$= Ar_o(2 - \sigma)\omega_s\cos\{(\sigma - 1)\omega_s t + \beta_1 - \gamma\} - Br_o(2 + \sigma)\omega_s\cos\{(\sigma + 1)\omega_s t + \beta_2 + \gamma\}$$

with

$$\gamma = \mathrm{Tan}^{-1}\{r_2/r_1\} \tag{5.47}$$

$$r_o = \sqrt{r_1^2 + r_2^2}\,. \tag{5.48}$$

5.6 Free (Nutation) Response to an Arbitrary Torque Impulse

In this section we consider the free response of a dual-spin vehicle to an impulse oriented in an arbitrary direction, i.e., it may have a spin axis (3-axis) component as well as the previously considered transverse component. The impulse is assumed small enough that linearized motion prevails. Arbitrary mass distribution of the platform is permitted while the rotor is assumed statically balanced.

We generate the torque impulse with an impulsive force $\mathbf{F}$ applied to the rotor at location $\mathbf{y}$ with respect to the rotor cm which in turn is located at $\mathbf{r}_1$ with respect to the spacecraft cm. $\mathbf{y}$ is fixed in rotor basis $\mathbf{e}_s$ and $\mathbf{r}_1$ is fixed in platform basis $\mathbf{e}_p$. Thus consider the torque

$$\mathbf{T} = [\mathbf{r}_1 + \mathbf{y}] \times \mathbf{F} \; . \tag{5.50}$$

The transverse components of $\mathbf{T}$ expressed in $\mathbf{e}_p$ are denoted T_1, T_2. The spin axis component of $\mathbf{y} \times \mathbf{F}$ adds spin momentum to the rotor, however, for linearized motion rotor spin rate is assumed fixed so this effect is neglected in the linearization process. The spin axis term of $\mathbf{r}_1 \times \mathbf{F}$ appears as an external torque perturbation on the platform as discussed on page 3.3 in connection with Eq. 3.16. Therefore, we denote the 3-axis term of $\mathbf{r}_1 \times \mathbf{F}$ as T_3, and the applied momentum impulse for torque pulse width τ is

$$\mathbf{M} = \mathbf{e}_p^T \tau [T_1, T_2, T_3]^T = \mathbf{e}_p^T [M_1, M_2, M_3]^T \; . \tag{5.51}$$

Applying this to plant dynamics of Eq. 2.40 and assuming the platform despun ($\omega_p = 0$) yields angular rates

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T [\omega_1, \omega_2, \omega_3]^T \tag{5.52}$$

$$= \mathbf{e}_p^T (1/\Delta') \begin{bmatrix} \{M_1[I_{22}I_{33}^p - I_{23}^2] + M_2[I_{12}I_{33}^p + I_{13}I_{23}] + M_3[I_{13}I_{22} + I_{12}I_{23}]\} \cos \lambda_p t \\ \qquad - \{[M_2 I_{33}^p + M_3 I_{23}] I_{11} \lambda_1/\lambda_p\} \sin \lambda_p t \\ \text{-- -- -- -- -- -- -- -- -- -- -- --} \\ \{M_1[I_{12}I_{33}^p + I_{13}I_{23}] + M_2[I_{11}I_{33}^p - I_{13}^2] + M_3[I_{23}I_{11} + I_{12}I_{13}]\} \cos \lambda_p t \\ \qquad + \{[M_1 I_{33}^p + M_3 I_{13}] I_{22} \lambda_2/\lambda_p\} \sin \lambda_p t \\ \text{-- -- -- -- -- -- -- -- -- -- -- --} \\ \{M_1[I_{13}I_{22} + I_{12}I_{23}] + M_2[I_{23}I_{11} + I_{12}I_{13}] + M_3[rI_{11}I_{22} - I_{12}^2]\} \cos \lambda_p t \\ \qquad + \{M_1[I_{23}I_{22}\lambda_2/\lambda_p] - M_2[I_{13}I_{11}\lambda_1/\lambda_p]\} \sin \lambda_p t + M_3 \Delta'/I_{33}^p \end{bmatrix}$$

where

$$\Delta' = I_{11} I_{22} I_{33}^p (1 - r) \tag{5.53}$$

$$\lambda_p^2 = \lambda_1 \lambda_2 / (1 - r) \; , \tag{5.54}$$

and λ_1, λ_2 are given by Eq. 2.35. r is given by Eq. 2.47e or more generally 1 - r is the s^3 coefficient of (2.40j) divided by $I_{11}I_{22}I_{33}^p$. The inertia elements I_{ij} are elements of $\hat{I}_p$ as defined in Eq. 3.15. Note if the platform is statically balanced M_3 above vanishes and the analysis of this section reduces to that of the preceding section. However, M_3 could be the result of an external force on the platform, in which case the analysis of this section applies irrespective of static balance.

The objective here is to determine the nutation response which is just Eq. 5.52. The nutation angle excursions are obtained by integrating (5.52), i.e., dividing by λ_p. Due to platform inertia asymmetry an asymmetric "pseudo ellipse" is traced out by the spin axis on a plane as in Figure 5.4 of the preceding section. Note, if we set $M_2 = M_3 = 0$ in (5.52) we again get the solution of that section.

For purposes of insight and to obtain a simple expression for nutation angle we make some simplifications in (5.52). Let $I_{12} = 0$ which can be effected by a new choice of $\mathbf{e}_p$ rotated about the 3-axis. Further, neglecting product terms in the imbalance inertias I_{13}, I_{23}, the rates reduce to

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T \begin{bmatrix} [M_1 + M_3(I_{13}/I_{33}^p)][1/I_{11}]\cos\lambda_p t - [M_2 + M_3(I_{23}/I_{33}^p)][(\lambda_1/\lambda_p)/I_{22}]\sin\lambda_p t \\ [M_2 + M_3(I_{23}/I_{33}^p)][1/I_{22}]\cos\lambda_p t + [M_1 + M_3(I_{13}/I_{33}^p)][(\lambda_2/\lambda_p)/I_{11}]\sin\lambda_p t \\ [M_1(I_{13}/I_{33}^p)/I_{11} + M_2(I_{23}/I_{33}^p)/I_{22}]\cos\lambda_p t + [M_1(I_{23}/I_{33}^p)(\lambda_2/\lambda_p)/I_{11} - M_2(I_{13}/I_{33}^p)(\lambda_1/\lambda_p)/I_{22}]\sin\lambda_p t + M_3/I_{33}^p \end{bmatrix}. \quad (5.55)$$

Finally, if we assume approximate symmetry, $I_{11} \approx I_{22} \approx I_T = \sqrt{I_{11}I_{22}}$, and define equivalent transverse momentum impulses

$$M_1' = M_1 + M_3 I_{13}/I_{33}^p \quad (5.56a)$$

$$M_2' = M_2 + M_3 I_{23}/I_{33}^p\ , \quad (5.56b)$$

we get

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T \begin{bmatrix} (M_1'/I_T)\cos\lambda_p t - (M_2'/I_T)\sin\lambda_p t \\ (M_2'/I_T)\cos\lambda_p t + (M_1'/I_T)\sin\lambda_p t \\ [M_1(I_{13}/I_{33}^p) + M_2(I_{23}/I_{33}^p)][1/I_T]\cos\lambda_p t + [M_1(I_{23}/I_{33}^p) - M_2(I_{13}/I_{33}^p)][1/I_T]\sin\lambda_p t + M_3/I_{33}^p \end{bmatrix}. \quad (5.57)$$

Then inspecting the transverse rate terms of (5.57) the symmetric approximation to nutation angle induced by the arbitrary torque impulse is

$$\theta_n = [M_1'^2 + M_2'^2]^{1/2}/[\lambda_p I_T] = M_T'/H\ . \quad (5.58)$$

Thus, for the symmetric case (and approximately otherwise) we get an equivalent transverse momentum impulse which yields the nutation angle in conventional form. The reader is cautioned to observe the special nature of M_3 discussed at the beginning of this section when this results from a force impulse on the rotor combined with platform static imbalance.

Next consider briefly the case where the initially despun platform is permitted to spin (open-loop) after application of torque impulse M_3. In this case for the symmetric vehicle the platform rates are

$$\boldsymbol{\omega}_p \approx \mathbf{e}_p^T \omega_p \begin{bmatrix} [(1 + \omega_p/\lambda_p)/I_T][I_{13}\cos\lambda_p t - I_{23}\sin\lambda_p t] - \omega_p I_{13}/H \\ [(1 + \omega_p/\lambda_p)/I_T][I_{23}\cos\lambda_p t + I_{13}\sin\lambda_p t] - \omega_p I_{23}/H \\ 1 \end{bmatrix}, \quad (5.59)$$

where the spin term is $\omega_p = M_3/I_{33}^p$ and H is the rotor spin momentum. This rate solution is the superposition of the impulse response from M_3 and the response to constant coning torques $-I_{23}\omega_p^{2,}$ and $I_{13}\omega_p^2$. The latter component is obtained by applying the constant torques to Eq. 2.40 with $\omega_p \neq 0$. The momentum is

$$\mathbf{H} \approx \mathbf{e}_p^T \begin{bmatrix} \omega_p(1 + \omega_p/\lambda_p)/[I_{13}\cos\lambda_p t - I_{23}\sin\lambda_p t - I_{13}] \\ \omega_p(1 + \omega_p/\lambda_p)/[I_{23}\cos\lambda_p t + I_{13}\sin\lambda_p t - I_{23}] \\ H + I_{33}^p\omega_p - \omega_p[(I_{13}^2 + I_{23}^2)/I_T][(1 + \omega_p/\lambda_p)\cos\lambda_p t - \omega_p/\lambda_p] \end{bmatrix}. \quad (5.60)$$

Eqs. 5.59 and 60 give the angular rate and momentum obtained previously for coning, but with nutation terms added. Note that the transverse momentum components vanish at t = 0 as they must for a spin torque impulse.

5.7 Small Angle Attitude and Spin Axis Motions Induced by Rotor Fixed External Torques

Consider a rotor fixed external torque step along the 1-axis expressed as

$$\mathbf{T} = \mathbf{e}_s^T T_1 [1,\ 0,\ 0]^T u(t) \tag{5.61}$$

$$= \mathbf{e}_p^T T_1 [\cos\omega_r t,\ \sin\omega_r t,\ 0]^T u(t)\ ,$$

where $\omega_r = \omega_s - \omega_p$ is rotor to platform relative rate. Let the rotor be balanced and symmetric and the platform balanced such that $I_{12} = I_{13} = I_{23} = 0$. Then inverting the transform of Eq. 2.40 with initial platform rate

$$\boldsymbol{\omega}_p(0) = \mathbf{e}_p^T [\omega_1(0),\ \omega_2(0),\ \omega_p]^T \tag{5.62}$$

yields

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T \begin{bmatrix} \omega_1(0)\cos\lambda_p t - \omega_2(0)(\lambda_1/\lambda_p)\sin\lambda_p t \\ \omega_1(0)(\lambda_2/\lambda_p)\sin\lambda_p t + \omega_1(0)\cos\lambda_p t \\ \omega_p \end{bmatrix} u(t) \tag{5.63}$$

$$+\ \mathbf{e}_p^T T_1/(\lambda_p^2 - \omega_r^2) \begin{bmatrix} [(\lambda_p^2 + \lambda_2\omega_r)/\lambda_p I_{11}]\sin\lambda_p t - [(\lambda_2 + \omega_r)/I_{11}]\sin\omega_r t \\ -[(\lambda_1 + \omega_r)/I_{22}]\cos\lambda_p t + [(\lambda_1 + \omega_r)/I_{22}]\cos\omega_r t \\ 0 \end{bmatrix} u(t)$$

where $\lambda_p^2 = \lambda_1\lambda_2$ with λ_1, λ_2 as expanded in Eq. 2.35. We have maintained the generality of platform asymmetry and steady spin rates of platform and rotor of $\omega_{p,}$ ω_s. The second term of 5.63 transforms to the rotor basis to give

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T T_1/[2(\lambda_p^2 - \omega_r^2)] \begin{bmatrix} [(\lambda_p^2 + \lambda_2\omega_r)/\lambda_p I_{11} + (\lambda_1 + \omega_r)/I_{22}]\sin(\lambda_p - \omega_r)t \\ +\ [(\lambda_p^2 + \lambda_2\omega_r)/\lambda_p I_{11} - (\lambda_1 + \omega_r)/I_{22}]\sin(\lambda_p + \omega_r)t \\ +\ [(\lambda_2 - \lambda_1)\omega_r/\lambda_1 I_{11}]\sin 2\omega_r t \\ - - - - - - - - - - - - - - - - - - - \\ -\ [(\lambda_p^2 + \lambda_2\omega_r)/\lambda_p I_{11} + (\lambda_1 + \omega_r)/I_{22}]\cos(\lambda_p - \omega_r)t \\ +\ [(\lambda_p^2 + \lambda_2\omega_r)/\lambda_p I_{11} - (\lambda_1 + \omega_r)/I_{22}]\cos(\lambda_p + \omega_r)t \\ +\ [(\lambda_2 - \lambda_1)\omega_r/\lambda_2 I_{22}]\cos 2\omega_r t + [(2\lambda_p^2 + (\lambda_2 + \lambda_1)\omega_r)/\lambda_2 I_{22}] \\ - - - - - - - - - - - - - - - - - - - \\ \omega_s \end{bmatrix}. \tag{5.64}$$

Note the presence of two frequencies, $\lambda_p - \omega_r$ and $\lambda_p + \omega_r$ in 5.64 due to asymmetry as obtained elsewhere herein for an impulse torque. The former is the usual rotor nutation frequency for a symmetric vehicle, $\lambda_s = \lambda_p - \omega_r = (\sigma_s - 1)\omega_s$, while the latter is $\lambda_p + \omega_r = (\sigma_s + 1)\omega_s$. Here we use rotor 'effective' inertia ratio as

$$\sigma_s = (I_s + I_p\omega_s/\omega_p)/I_T\ . \tag{5.65}$$

The remaining goals of this section are adequately served by assuming symmetry, i.e., $I_{11} = I_{22} = I_T$, which also gives $\lambda_p = \lambda_1 = \lambda_2$. The symmetric vehicle rate solution is generalized to a pulse torque however. The pulse response is useful for thruster nutation control analysis. Specifically the torque pulse is written

$$\mathbf{T} = \mathbf{e}_s^T T_1 [1,\ 0,\ 0]^T [u(t) - u(t - t_1)] \tag{5.66}$$

$$= \mathbf{e}_p^T T_1 [\cos\omega_r t,\ \sin\omega_r t,\ 0]^T [u(t) - u(t - t_1)]\ .$$

The symmetric balanced vehicle platform rate solution is

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T \begin{bmatrix} \omega_1(0)\cos\lambda_p t - \omega_2(0)\sin\lambda_p t \\ \omega_1(0)\sin\lambda_p t + \omega_2(0)\cos\lambda_p t \\ \omega_p \end{bmatrix} u(t) + \mathbf{e}_p^T [T_1/I_T\lambda_s] \begin{bmatrix} \sin\lambda_p t - \sin\omega_r t \\ -\cos\lambda_p t + \cos\omega_r t \\ 0 \end{bmatrix} u(t) \tag{5.67}$$

$$+ \mathbf{e}_p^T [T_1/I_T\lambda_s] \begin{bmatrix} -\sin\{\lambda_p t - \lambda_s t_1\} + \sin\omega_r t \\ \cos\{\lambda_p t - \lambda_s t_1\} - \cos\omega_r t \\ 0 \end{bmatrix} u(t - t_1) \ .$$

The corresponding rotor rate is

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T \begin{bmatrix} \omega_1(0)\cos\lambda_s t - \omega_2(0)\sin\lambda_s t \\ \omega_1(0)\sin\lambda_s t + \omega_2(0)\cos\lambda_s t \\ \omega_s \end{bmatrix} u(t) + \mathbf{e}_s^T [T_1/I_T\lambda_s] \begin{bmatrix} \sin\lambda_s t \\ -\cos\lambda_s t \\ 0 \end{bmatrix} u(t) \tag{5.68}$$

$$+ \mathbf{e}_s^T [T_1/I_T\lambda_s] \begin{bmatrix} -\sin\lambda_s(t - t_1) \\ \cos\lambda_s(t - t_1) \\ 0 \end{bmatrix} u(t - t_1) + \mathbf{e}_s^T [T_1/I_T\lambda_s] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [u(t) - u(t - t_1)] \ .$$

Summarizing, 5.67 and 5.68 give the angular rate responses of rotor and platform in their respective vector bases due to a rotor fixed torque pulse for the symmetric balanced dual-spin vehicle with rotor and platform spinning respectively at rates ω_s, ω_p. Inertial, platform, and rotor nutation frequencies are respectively

$$\lambda_o = \sigma_s\omega_s = H/I_T \tag{5.69a}$$

$$\lambda_p = \lambda_o - \omega_p = \sigma_s\omega_s - \omega_p \tag{5.69b}$$

$$\lambda_s = \lambda_o - \omega_s = \sigma_s\omega_s - \omega_s = (\sigma_s - 1)\omega_s \ . \tag{5.69c}$$

Next we shall consider the approximate small angle inertial motion of the spin axis and momentum vector with a rotor fixed step torque. Let $\mathbf{e}_i$ be a despun basis approximately inertially fixed which is related to the platform basis as $\mathbf{e}_i = B(\omega_p t)^T \mathbf{e}_p$ where B is given by Eq. 1.10. Transforming the platform transverse rates to the inertial basis

$$\omega_i = \begin{bmatrix} \omega_1(0)\cos\lambda_o t - \omega_2(0)\sin\lambda_o t \\ \omega_1(0)\sin\lambda_o t + \omega_2(0)\cos\lambda_o t \end{bmatrix} + \frac{T_1}{I_T\lambda_s} \begin{bmatrix} \sin\lambda_o t \\ -\cos\lambda_o t \end{bmatrix} u(t) \tag{5.70}$$

$$+ \frac{T_1}{I_T\lambda_s} \begin{bmatrix} -\sin\{\lambda_o t - \lambda_s t_1\} \\ \cos\{\lambda_o t - \lambda_s t_1\} \end{bmatrix} u(t - t_1) + \frac{T_1}{I_T\lambda_s} \begin{bmatrix} -\sin\omega_s t \\ \cos\omega_s t \end{bmatrix} [u(t) - u(t - t_1)]$$

$$= \begin{bmatrix} \omega_1(0)\cos\lambda_o t - \omega_2(0)\sin\lambda_o t \\ \omega_1(0)\sin\lambda_o t + \omega_2(0)\cos\lambda_o t \end{bmatrix} + \frac{T_1}{I_T\lambda_s} 2\sin[(\lambda_o - \omega_s)t/2] \begin{bmatrix} \cos[(\lambda_o + \omega_s)t/2] \\ \sin[(\lambda_o + \omega_s)t/2] \end{bmatrix} u(t)$$

$$+ \frac{T_1}{I_T\lambda_s} \begin{bmatrix} -\sin\{\lambda_o t - \lambda_s t_1\} - \sin\omega_s t \\ \cos\{\lambda_o t - \lambda_s t_1\} + \cos\omega_s t \end{bmatrix} u(t - t_1) \ .$$

The latter form of 5.70 is sometimes a more easily recognized form of the rate signal in a simulation, i.e., a signal of frequency $(\lambda_o + \omega_s)/2 = (\sigma_s + 1)\omega_s/2$ with a modulating envelope at frequency $(\lambda_o - \omega_s)/2 = (\sigma_s - 1)\omega_s/2$. When the platform is despun 5.70 is the platform rate. Integrating the step portion of 5.70 ($t_1 \to \infty$) for small angle motion

$$\boldsymbol{\theta}_i = \boldsymbol{\theta}_i(0) + 1/\lambda_o\begin{bmatrix}\omega_1(0)\sin\lambda_o t + \omega_2(0)[\cos\lambda_o t - 1]\\ \omega_2(0)\sin\lambda_o t - \omega_1(0)[\cos\lambda_o t - 1]\end{bmatrix} + [T_1/I_T\lambda_s]\begin{bmatrix}[1-\cos\lambda_o t]/\lambda_o - [1-\cos\omega_s t]/\omega_s\\ -[\sin\lambda_o t]/\lambda_o + [\sin\omega_s t]/\omega_s\end{bmatrix}u(t) \qquad (5.71)$$

$$= \boldsymbol{\theta}_i(0) + 1/\lambda_o\begin{bmatrix}\omega_1(0)\sin\lambda_o t + \omega_2(0)[\cos\lambda_o t - 1]\\ \omega_2(0)\sin\lambda_o t - \omega_1(0)[\cos\lambda_o t - 1]\end{bmatrix} + [T_1/H\lambda_s]\left\{\begin{bmatrix}1-\sigma_s\\ 0\end{bmatrix} - \begin{bmatrix}\cos\lambda_o t\\ \sin\lambda_o t\end{bmatrix} + \sigma_s\begin{bmatrix}\cos\omega_s t\\ \sin\omega_s t\end{bmatrix}\right\}u(t)$$

where here the latter form is written for easier visualization of the motion. Depicted on Figure 5.5, this motion can be viewed as the sum of a constant plus two vectors rotating respectively at rates λ_o and ω_s. On the figure labels the substitutions $\lambda_o = \omega_n$ and $\sigma_s = \sigma$ are used.

Concurrently, the momentum vector angle in inertial space is

$$\boldsymbol{\phi}(t) = \boldsymbol{\phi}(0) + [1/H]\int_0^t T(t)dt = \phi(0) + [T/H\omega_s]\begin{bmatrix}\cos\omega_s t - 1\\ \sin\omega_s t\end{bmatrix}u(t) \qquad (5.72)$$

$$= \boldsymbol{\phi}(0) + [2T/H\omega_s]\sin[\omega_s t/2]\begin{bmatrix}\sin[\omega_s t/2]\\ \cos[\omega_s t/2]\end{bmatrix}u(t)\ ,$$

which, as depicted, is the sum of a constant and a term rotating at spin rate ω_s.

The response of this section can be used to predict behavior under numerous circumstances where rotor fixed torques are applied. One source of a rotor fixed torque is the pulse from an ideal apogee boost motor which has misalignment and offset errors producing mispointing and reduction in the delivered velocity impulse (see Figure 5.6). We use this as an example for application of the geometry of Figure 5.5. The maximum nutation angle (between momentum vector and spin axis) is seen to occur when the momentum vector is at the origin and the spin axis is at it's maximum excursion along the 2-axis of Figure 5.5. This maximum nutation is $2T/\{H\omega_s|1-\sigma_s|\}$. Attitude error is the instantaneous excursion of the momentum vector from its initial position and is bounded by $2T/H\omega_s$. Considering the average spin axis position over a torque pulse of many spin and nutation cycles duration, it is noted that θ_1, θ_2 have average values $T/H\omega_s$, and zero respectively. Thus, on the average the boost acceleration has a pointing error of $T/H\omega_s$. Finally consider the thrust loss due to coning. The coning motion produces a spinning transverse acceleration which integrates to zero net velocity change and hence represents a reduction of the motor delivered impulse. If the thrust vector were offset by a fixed angle from the spin axis, then the velocity impulse lost ΔV is related to the net impulse delivered V_o by $\Delta V/V_o = 1 - \cos\delta \approx \delta^2/2$. V_o is delivered along the direction $\bar{\theta}_i = (-T/H\omega_s, 0)$, while it may be observed that δ between this average direction and the instantaneous thrust varies from $\min\{\delta\} = T/H\omega_s$ to $\max\{\delta\} = (T/H\omega_s)(1+\sigma_s)/(1-\sigma_s)$ with a mean value $\delta_o = (T/H\omega_s)/(1-\sigma_s)$. One approach to thrust loss estimation (suggested by D. Challoner) is

$$\Delta V/V_o = 1 - \cos\delta \approx 1 - (1 - \delta^2/2) = \delta^2/2\ . \qquad (5.73)$$

Then using 5.72 and manipulating somewhat

$$\delta^2 = |\theta_i(t) - \bar{\theta}_i|^2 = [T/\{H\omega_s(\sigma_s - 1)\}]^2[1 + \sigma_s^2 - 2\sigma_s\cos\{(\lambda_o - \omega_s)t\}]\ , \qquad (5.74)$$

such that

$$\bar{\delta}^2 = (1+\sigma_s^2)[T/\{H\omega_s(\sigma_s - 1)\}]^2\ . \qquad (5.75)$$

Although this probably doesn't give the loss for any real situation it probably does give the functional dependence on important parameters. Summarizing the apogee boost error bounds are approximated:

Maximum Nutation : $2T/H\omega_s|1-\sigma_s| = 2T/H|\lambda_s| = 2\rho$ (5.76a)

Maximum Attitude Error : $2T/H\omega_s$ (5.76b)

Average Thrust Pointing Error : $T/H\omega_s$ (5.76c)

Percent Coning Loss : $\Delta V/V_o = \{(1+\sigma_s^2)/2\}[T/\{H\omega_s(\sigma_s - 1)\}]^2$ (5.76d)

$$= \{(1+\sigma_s^2)/2\}[T/\{H\lambda_s\}]^2 = \{(1+\sigma_s^2)/2\}\rho^2\ .$$

Percent Coning Loss(Imbalance) : $= \{\sigma_s^2/2\}[T/\{H\omega_s(\sigma_s - 1)\}]^2 = \{\sigma_s^2/2\}[T/\{H\lambda_s\}]^2 = \{\sigma_s^2/2\}\rho^2$. (5.76e)

A detailed expansion of the coning loss with simultaneous disturbance torque and dynamic imbalance in both axes provided by Jack Murphy is recorded here as

$$\frac{\Delta V}{V_o} = 1 - \cos\bar{\xi} \approx \frac{\bar{\xi}^2}{2}$$

$$\approx \frac{1}{2}\left\{\left[1+\frac{1}{\sigma^2}\right]\left\{\left[\frac{\sigma_2 T_1}{H\omega_s(\sigma_2-1)}\right]^2+\left[\frac{\sigma_1 T_2}{H\omega_s(\sigma_1-1)}\right]^2\right\}+T_1 I_{23}\left[\frac{\sigma_2}{[H(\sigma_2-1)]}\right]^2-T_2 I_{13}\left[\frac{\sigma_1}{[H(\sigma_1-1)]}\right]^2+\left[\frac{\sigma_1 I_{13}}{I_s(\sigma_1-1)}\right]^2+\left[\frac{\sigma_2 I_{23}}{I_s(\sigma_2-1)}\right]^2\right\}.$$

If the transverse torque is known, and the engine is restartable, a simple two-burn profile can be used to minimize the average thrust pointing error, θ_e, and reduce the thrust impulse perturbation ΔV_e (Ref. 33).

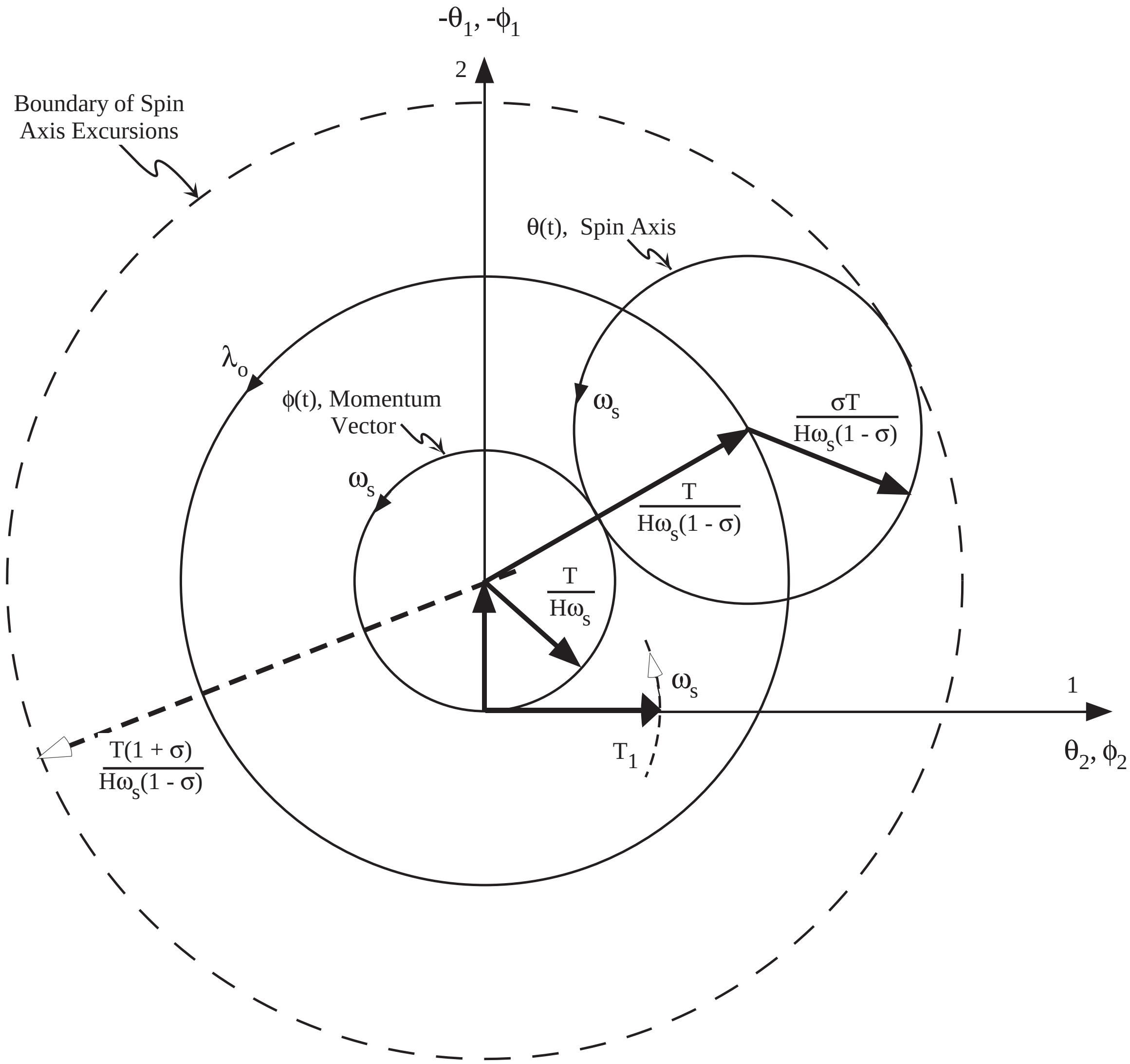


Figure 5.5 Spin Axis and Momentum Vector Motion in Inertial Space Under Constant Rotor Fixed Torque Without Nutation Damping.

Constant Rotor Fixed Torque Without Nutation Damping.

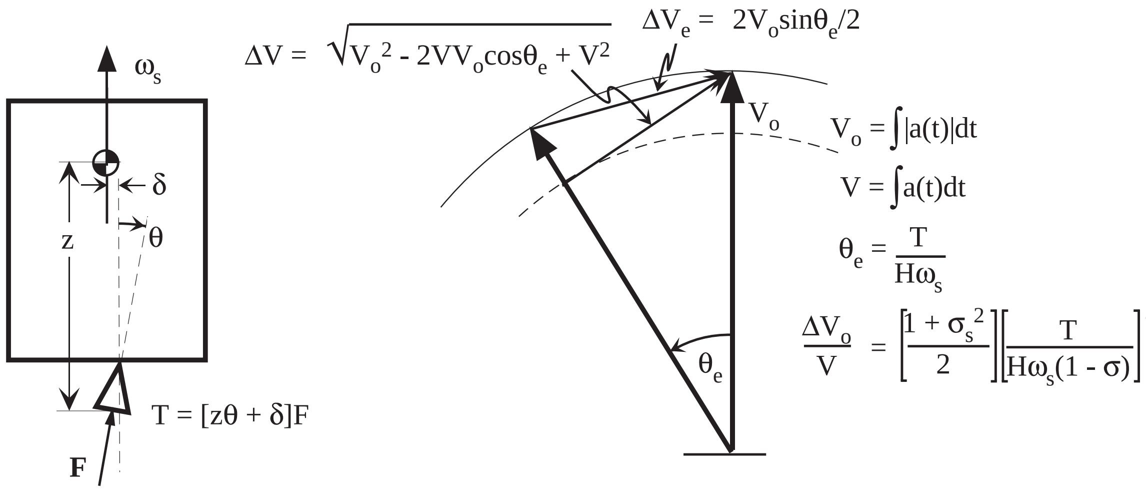


Figure 5.6 Velocity Impulse Error Definitions.

Transverse Inertia Asymmetry and Dynamic Imbalance Spin Up/Down

A rotor fixed transverse torque in the presence of rotor transverse inertia asymmetry can produce a secular spin torque (Ref. 4). Let the rotor transverse inertia asymmetry be denoted $\Delta I_T^s = I_{22}^s - I_{11}^s$. Then the spin torque may be written

$$T_3 = -\Delta I_T^s \omega_{10}\omega_{20} - I_{13}^s \omega_{20}\omega_s + I_{23}^s \omega_{10}\omega_s \,.$$

The asymmetry component is maximized when the transverse rate is equally distributed between the two axes at $T_3 = [\Delta I_T^s/2]\omega_T^2$. Using the transverse rate due to an open-loop transverse disturbance torque T_1 from Eq. 5.68

$$T_3 = \frac{\Delta I_T^s}{2}\,\omega_T^2 = \frac{\Delta I_T^s}{2}\left[\frac{T_1}{I_T\lambda_s}\right]^2 = \frac{\Delta I_T^s}{2}\left[\frac{T_1}{I_T(\sigma-1)}\right]^2 \frac{1}{\omega_s^2} = I_s\dot{\omega}_s = I_s\,\frac{d\omega_s}{dt} \,.$$

Integrating,

$$\omega_s^3 = \omega_{so}^3 + \frac{\Delta I_T^s}{2I_s}\left[\frac{T_1}{I_T(\sigma-1)}\right]^2 t \,.$$

For the imbalance effect, using the transverse rate solutions from 5.68

$$T_3 = \left[\frac{-I_{13}^s T_1 + I_{23}^s T_2}{I_T}\right]\frac{\omega_s}{\lambda_s} = \left[\frac{-I_{13}^s T_1 + I_{23}^s T_2}{I_T(\sigma-1)}\right] = \theta_{w2}T_1 + \theta_{w1}T_2 \,,$$

so the asymmetry spin acceleration is proportional to $1/\omega_s^2$ while the imbalance acceleration is constant. Note that the transverse rate due to the imbalance itself, $\theta_w\omega_s$ does not couple because it is in the same plane as the imbalance. However, as the last form shows, the transverse rate cross coupled torque is exactly that which results by viewing the torque in the *principal basis*.

Dead-Beat Thrust Startup

When a velocity change is executed with a thruster having a large transverse torque, for example a radially offset axial thruster, the coning and nutation rates and angles become undesirably large inducing a large thrust loss and/or pointing error. By using a properly timed startup pulse it is possible to execute the maneuver without nutation. From 5.68 the nutation during thrust is normally

$$\boldsymbol{\omega}_s(t) = \rho\lambda_o \mathbf{e}_s^T[\sin\lambda_s t, -\cos\lambda_s t, 0]^T$$

so the technique is to establish an initial condition with the startup pulse that cancels this steady state nutation. Beginning with no nutation, firing a one sixth nutation period duration pulse, $\lambda_s t_1 = \pi/3$, produces rate

$$\boldsymbol{\omega}_s(t) = (\rho\lambda_o/2)\mathbf{e}_s^T[\sin\lambda_s t + \sqrt{3}\cos\lambda_s t,\ -\cos\lambda_s t + \sqrt{3}\sin\lambda_s t,\ 0]^T$$

$$= \rho\lambda_o\mathbf{e}_s^T[\sin(\lambda_s t + \pi/3),\ -\cos(\lambda_s t + \pi/3),\ 0]^T\ .$$

Then initiating continuous thrust at $\lambda_s t_2 = 2\pi/3 + 2n\pi = (1 + 3n)(2\pi/3)$ will induce nutation that cancels the initial condition resulting in no nutation during the continuous burn.

The thrust pointing error can be minimized by positioning the average spin axis position in the desired thrust direction. This can be implemented by initially offsetting the attitude in an arbitrary direction of magnitude equal to the cone angle, then starting the maneuver when the transverse torque vector is inertially aligned with the attitude offset rotation vector. When combined with the deadbeat it necessary that $t_2 - t_1$ be equal to an integer number, say m, of spin periods to preserve the pre-established *average attitude* position. This leads to the relation

$$t_2 - t_1 = \frac{1}{\lambda_s}\left[\frac{\pi}{3} + 2n\pi\right] = \frac{2\pi}{\lambda_s}\left[\frac{1}{6} + n\right] = m\,\frac{2\pi}{\omega_s}\ ,$$

giving nominally

$$n = |\sigma - 1|m - 1/6\ .$$

5.8 Small Angle Spin Axis Motion Induced by a Despun Platform Fixed External Torque

Assume initial platform angular rates

$$\boldsymbol{\omega}_p(0) = \mathbf{e}_p^T \omega_p(0) = \mathbf{e}_p^T [\omega_1(0),\ \omega_2(0),\ \omega_3(0)]^T \tag{5.77}$$

and a step torque

$$\mathbf{T} = \mathbf{e}_p^T [T_1,\ 0,\ 0]^T u(t) \tag{5.78}$$

applied at time zero. We wish to determine the spin axis and angular momentum vector motion given that $\mathbf{T}$ is small enough that linear motion prevails. We first invert Eq. 2.40 to obtain the free response to the initial condition, or equivalently to an impulse input, $\mathbf{T} = \mathbf{e}_p^T I_p \omega_p(0)\delta(t)$. The result is

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T [\omega_{p1},\ \omega_{p2},\ \omega_{p3}]^T\ , \tag{5.79a}$$

with

$$\Delta'\omega_{p1} = \Big\{[I_{22}I_{33}^p - I_{23}^2]I_{11} - [I_{12}I_{33}^p + I_{13}I_{23}]I_{12} - [I_{13}I_{22} + I_{12}I_{23}]I_{13}\Big\}\omega_1(0)\cos\lambda_p t \tag{5.79b}$$

$$- \Big\{\{\Delta'\lambda_1/[I_{22}\lambda_p(1-r)]\}[-I_{12}\omega_1(0) + I_{22}\omega_2(0) - I_{23}\omega_3(0)] - I_{23}I_{11}(\lambda_1/\lambda_p)[-I_{13}\omega_1(0) - I_{23}\omega_2(0) + I_{33}^p\omega_3(0)]\Big\}\sin\lambda_p t$$

$$\Delta'\omega_{p2} = \Big\{[I_{11}I_{33}^p - I_{13}^2]I_{22} - [I_{12}I_{33}^p + I_{13}I_{23}]I_{12} - [I_{23}I_{11} + I_{12}I_{13}]I_{23}\Big\}\omega_2(0)\cos\lambda_p t \tag{5.79c}$$

$$+ \Big\{\{\Delta'\lambda_2/[I_{11}\lambda_p(1-r)]\}[I_{11}\omega_1(0) - I_{12}\omega_2(0) - I_{13}\omega_3(0)] + I_{13}I_{22}(\lambda_2/\lambda_p)[-I_{13}\omega_1(0) - I_{23}\omega_2(0) + I_{33}^p\omega_3(0)]\Big\}\sin\lambda_p t$$

$$\Delta'\omega_{p3} = \Big\{[I_{11}I_{22}\lambda_1\lambda_2/\lambda_p^2][I_{13}\omega_1(0) + I_{23}\omega_2(0)]\Big\}[\cos\lambda_p t - 1] + [I_{11}I_{22}\lambda_1\lambda_2/\lambda_p^2]I_{33}^p\omega_3(0) \tag{5.79d}$$

$$+ \Big\{[I_{12}I_{13}I_{11}\lambda_1 + I_{23}I_{11}I_{22}\lambda_2](\omega_1(0)/\lambda_p) - [I_{12}I_{23}I_{22}\lambda_2 + I_{13}I_{22}I_{11}\lambda_1](\omega_2(0)/\lambda_p)\Big\}\sin\lambda_p t$$

where λ_p is platform nutation frequency,

$$\Delta' = I_{11}I_{22}I_{33}^p(1-r) \tag{5.80}$$

and r is defined by noting that Δ' is the coefficient of s^3 in $\Delta(s)$ given in Eq. 2.40j. Also note that with the platform despun, $\lambda_i/\lambda_p = \sqrt{I_{jj}/I_{ii}}$, and $I_{ii}\lambda_i/\lambda_p = I_T = \sqrt{I_{11}I_{22}}$. The secular 3-axis term will vanish if there is any position control. If the platform is assumed balanced and symmetric the 3-axis term above vanishes and the remaining coefficients reduce to simply the rate initial values. Next we invert (2.40), as in (2.43), to get the driven solution due to step torque T_1. This operation gives

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T T_1/(\lambda_p\Delta')\begin{bmatrix} [I_{22}I_{33}^p - I_{23}^2]\sin\lambda_p t \\ [I_{12}I_{33}^p + I_{13}I_{23}]\sin\lambda_p t + I_{33}^p I_{22}(\lambda_2/\lambda_p)[1-\cos\lambda_p t] \\ [I_{13}I_{22} + I_{12}I_{23}]\sin\lambda_p t + I_{23}I_{22}(\lambda_2/\lambda_p)[1-\cos\lambda_p t] \end{bmatrix}. \tag{5.81}$$

To obtain the spin axis motion we integrate, dropping the spin axis term as it is assumed a despin control system maintains the platform despun. Hence,

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \mathbf{e}_p^T T_1/(\lambda_p^2\Delta')\begin{bmatrix} [I_{22}I_{33}^p - I_{23}^2][1-\cos\lambda_p t] \\ [I_{12}I_{33}^p + I_{13}I_{23}][1-\cos\lambda_p t] + I_{33}^p I_{22}(\lambda_2/\lambda_p)[\lambda_p t - \sin\lambda_p t] \\ 0 \end{bmatrix}. \tag{5.82}$$

Simplifying to the balanced case $I_{13} = I_{23} = 0$, and $I_{12} = 0$ as well,

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \mathbf{e}_p^T T_1/(I_{11}\lambda_p^2)[1-\cos\lambda_p t,\ \lambda_2 t - \sin\lambda_p t,\ 0]^T\ . \tag{5.83}$$

The motion is shown on Figure 5.7 where the notation $\lambda_p = \omega_n$ is used. The corresponding attitude (momentum vector) motion is

$$\phi(t) = \phi(0) + (1/H)\int T_1(t)dt = \phi(0) + t\,T_1/H = \phi(0) + t\,T_1/I_s\omega_s \tag{5.84}$$

and this motion is also shown on Figure 5.7. It is simple to show that $\theta_2(t) = \phi_2(t)$ every half nutation period, and $\theta(t) = \phi(t)$, i.e., the spin axis and momentum vector are coincident every full nutation period.

It is of interest also to know the angular rates in the rotor basis due to the platform fixed (inertial) constant torque. Using again the balanced platform representation and transforming to the rotor basis

$$\boldsymbol{\omega}_s = \mathbf{e}_s^T T_1/(I_{11}\lambda_p)\begin{bmatrix} \sin\lambda_s t + (\lambda_2/\lambda_p)\sin\omega_r t \\ -\cos\lambda_s t + (\lambda_2/\lambda_p)\cos\omega_r t \\ 0 \end{bmatrix}, \tag{5.85}$$

where ω_r is rotor to platform relative rate and $\lambda_s = \lambda_p - \omega_r$ is rotor nutation frequency. Thus, both the anticipated nutation and precession terms are evident.

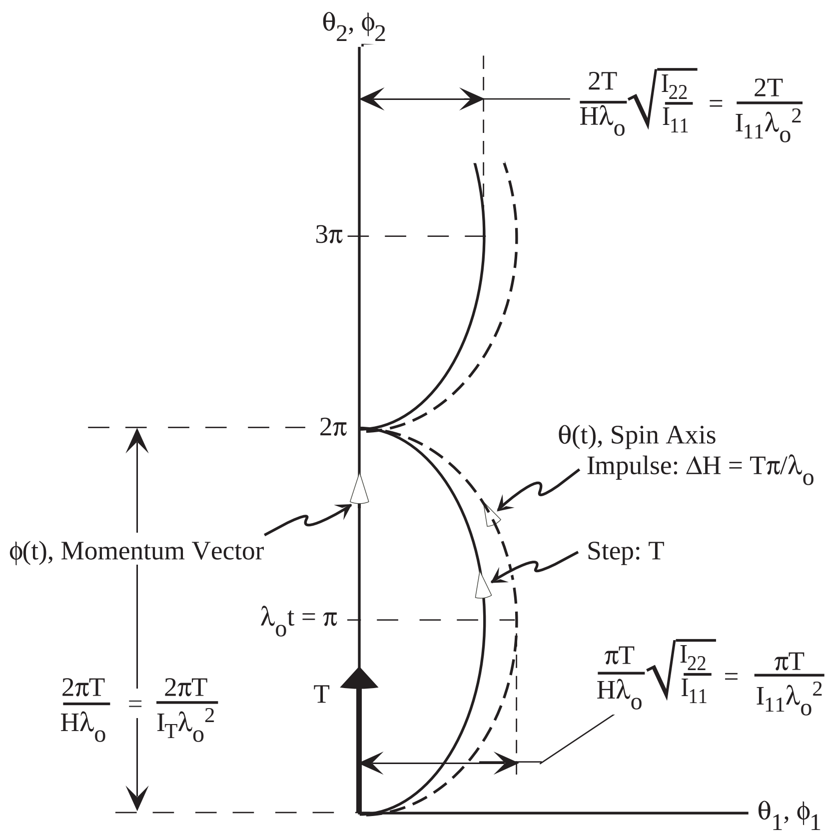


Figure 5.7 Spin Axis and Momentum Vector Small Angle Motion in Inertial Space Under Impulse and Constant Despun Platform Fixed (Inertial) Torque.

Space Under Impulse and Constant Despun Platform Fixed Torque.

Nutation Free Precession Maneuver

Given the necessary torqueing capability, it would be desirable to perform a pure precession maneuver without nutation. For this purpose consider the simple balanced case (all products of inertia zero) and allowing nonzero initial conditions, sum 5.79 and 5.81 to get

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T \left\{ T_1 I_{22} I_{33}^p/(\lambda_p \Delta') \begin{bmatrix} \sin\lambda_p t \\ (\lambda_2/\lambda_p)[1-\cos\lambda_p t] \\ 0 \end{bmatrix} + \begin{bmatrix} \omega_1(0)\cos\lambda_p t - (\lambda_1/\lambda_p)\omega_2(0)\sin\lambda_p t \\ \omega_2(0)\cos\lambda_p t - (\lambda_2/\lambda_p)\omega_1(0)\sin\lambda_p t \\ 0 \end{bmatrix} \right\} . \tag{5.86}$$

Setting $\omega_1(0) = \omega_3(0) = 0$ and $\omega_2(0) = T_1/I_{11}\lambda_1 = T_1/H$ yields the desired pure precession over the interval t, t_1, as

$$\omega_2(t) = T_1/H[u(t) - u(t - t_1)]$$

when the control torque is

$$\mathbf{T} = \mathbf{e}_p^T\{T_1[u(t) - u(t - t_1)],\ I_{22}T_1/H[\delta(t) - \delta(t - t_1)],\ 0\}^T . \tag{5.87}$$

The initial impulse torque establishes a nutation corresponding to 5.79 that cancels the nutation induced by application of the step precession torque of 5.81. Removal of the step at time t_1 induces a new nutation that is canceled by the additional negative impulse at the same instant. For the general unbalanced case, when one equates coefficients of $\sin\lambda t$, $\cos\lambda t$ in 5.79 and 5.81, six equations in three unknowns result. Thus, it does not appear there is a solution of this form that gives the nutation free precession maneuver.

Spinning Spacecraft Wobble Capture Maneuver

Consider a single body spinning spacecraft spinning about its maximum principle axis of inertia, which is displaced from a desired or geometric spin axis by a dynamic imbalance product of inertia I_{13}. The maneuver investigated here, subject of a U. S. Patent (Ref. 32), is to move the spin axis into the inertial position of the principal axis, and the angular momentum vector H, and arrest its coning motion. Initially the H vector is inertially fixed, with no torque on the vehicle, and the spin axis cones at spin rate about it at the wobble angle $\theta_w = I_{13}/[I_T - I_s]$. After the capture the spin axis is fixed in exactly the direction initially describing the momentum vector H, and the H vector cones at spin rate about this line with cone angle $\theta_H = T/H\omega_s = I_{13}\omega_s^2/I_s\omega_s^2 = I_{13}/I_s$. The capture scenario is sketched on Figure 5.8 below. Observe the ratio of momentum coning angle to wobble angle is $\theta_H/\theta_w = [I_T - I_s]/I_s = [1 - \sigma]/\sigma \approx 1 - \sigma$, which is small when the inertia ratio is near unity.

With no nutation prior to control application, $\omega(t) = [\theta_w\omega_s, 0, 0]^T$. If at time t = 0 we apply a step torque $T_2 = -I_{13}\omega_s^2$, (see for example the right side of 2.34b) the initial wobble rate becomes an initial nutation rate $\omega_1(t) = \theta_w\omega_s$. Now the objective is to null the nutation while driving the spin axis to the position of the initial momentum vector. With adequate sensing, one might apply control feedback torque to simultaneously null nutation rates and spin axis position error. Alternatively, we might null rates in a manner that approximately preserves attitude. One simple open-loop solution that is helpful in underatanding the problem is to apply a sinusoidal transverse nutation damping torque in the body at body nutation frequency $\hat{\lambda}_s^2 = [\lambda_s^2 + r(3\lambda_s\omega_s + 2\omega_s^2)]/(1 - r)$. Such a torque is expressed

$$\mathbf{T} = \mathbf{e}_s^T \begin{bmatrix} T_1\cos\hat{\lambda}_s t \\ T_2\sin\hat{\lambda}_s t \\ 0 \end{bmatrix} = \mathbf{e}_i^T \begin{bmatrix} T_1\cos\psi\cos\hat{\lambda}_s t - T_2\sin\psi\sin\hat{\lambda}_s t \\ T_2\cos\psi\sin\hat{\lambda}_s t + T_1\sin\psi\cos\hat{\lambda}_s t \\ 0 \end{bmatrix} = \mathbf{e}_i^T \begin{bmatrix} T_1\cos\omega_s t\cos\hat{\lambda}_s t - T_2\sin\omega_s t\sin\hat{\lambda}_s t \\ T_2\cos\omega_s t\sin\hat{\lambda}_s t + T_1\sin\omega_s t\cos\hat{\lambda}_s t \\ 0 \end{bmatrix} \tag{5.88}$$

$$\approx \mathbf{e}_i^T\,\frac{1}{2}\begin{bmatrix} T_1\cos\lambda_o t + T_2\cos\lambda_o t \\ T_2\sin\lambda_o t + T_1\sin\lambda_o t \\ 0 \end{bmatrix}, \approx \mathbf{e}_i^T T_o \begin{bmatrix} \cos\lambda_o t \\ \sin\lambda_o t \\ 0 \end{bmatrix},$$

the latter expression for $T_o = T_1 = T_2$. One method to solve for the response is to substitude ω_s for ω_p in the required terms of (2.40). The resultant transverse rates due to this torque approximate

$$\omega(t) \approx \frac{t}{2(1-r)I_T}\begin{bmatrix}[T_1 + (1-r)T_2]\cos\hat{\lambda}_s t \\ [T_2 + (1-r)T_1]\sin\hat{\lambda}_s t\end{bmatrix} \approx \frac{T_o t}{I_T}\begin{bmatrix}\cos\hat{\lambda}_s t \\ \sin\hat{\lambda}_s t\end{bmatrix}. \qquad (5.89)$$

This sinusoidal torque can simply be applied for enough time to allow the driven nutation frequency rates to cancel the nutation, i.e., $t = \omega_1(0)I_T/T_o$, and then removed while the above mentioned step torque remains on to cancel the wobble dynamic torque. If the nutation period is long in comparison to the spin period, the nutation damping sinusoidal torque will not change significantly over a spin period, and will not produce a large attitude perturbation. The attitude error can be more quantitatively bounded by integrating the torque of Eq. 5.

$$\theta_a(t) = \ = \mathbf{e}_i^T \frac{1}{H}\int_0^t \mathbf{T}dt = \mathbf{e}_i^T \frac{T_o}{H}\int_0^t \begin{bmatrix}\cos\lambda_o t \\ \sin\lambda_o t \\ 0\end{bmatrix}dt = \mathbf{e}_i^T \frac{T_o}{H\lambda_o}\begin{bmatrix}\sin\lambda_o t \\ 1-\cos\lambda_o t \\ 0\end{bmatrix}. \qquad (5.90)$$

From this expression we see as expected that the attitude excursion due to the nutation damping sinusoid is oscillatory at inertial nutation frequency. Suppose we choose to damp the initial nutation in n body nutation periods. Using $t = 2n\pi/\lambda_s$, and equating the coefficient of (5.89) to the initial rate $\omega_1(0) = \theta_w\omega_s$, gives $T_o = I_{13}\omega_s^2/[2n\pi]$. The bound on attitude excursion is max $\{\theta_a\} = 2T_o/[H\lambda_o] = I_{13}/[n\pi\sigma I_s] \approx I_{13}/[n\pi I_s]$. The implication that making n large will arbitrarily reduce attitude error will not be correct for large wobble angles where the linearization becomes weaker.

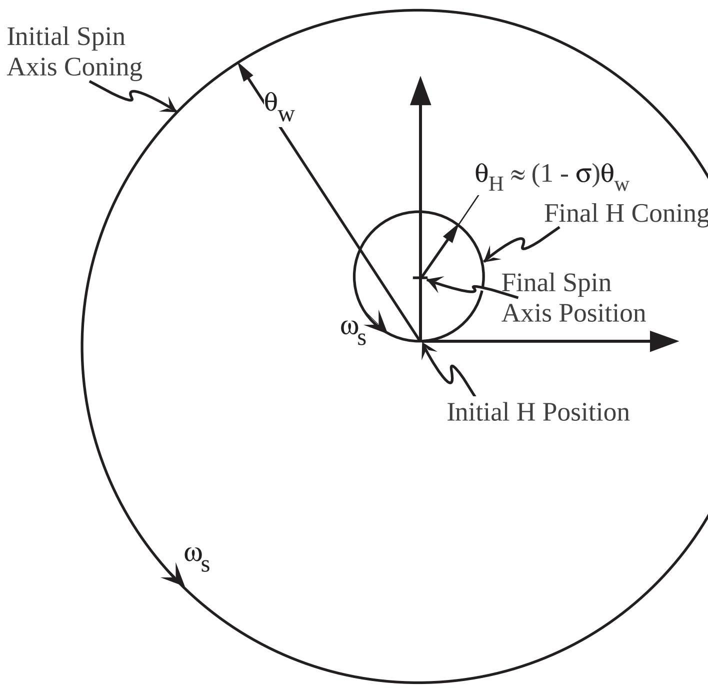


Figure 5.8 Wobble Capture Geometry.

5.9 Nutation Induced by a Uniform Transverse Torque Impulse Series

Uniform Pulses

Consider a series of torque impulses of magnitude θ_o applied at period Δt and all in a fixed direction in a vector basis from which nutation frequency is observed at λ. Two of the most frequently encountered cases are inertially fixed pulses with $\lambda = \lambda_o = \sigma\omega_s$, or body fixed pulses where $\lambda = \lambda_s = (\sigma - 1)\omega_s$. Let the nutation damping time constant be τ. Then the nutation immediately following the nth pulse is

$$\theta_1 = \theta_o$$

$$\theta_2 = \theta_1 e^{-(\Delta t/\tau - j\lambda\Delta t)} + \theta_o$$

$$\theta_n = \theta_{n-1} e^{-(\Delta t/\tau - j\lambda\Delta t)} + \theta_o \tag{5.91}$$

$$= \theta_o \sum_{i=1}^{n} e^{-(i-1)(\Delta t/\tau - j\lambda\Delta t)}$$

$$= \theta_o \sum_{i=1}^{n} x^{(i-1)} = \theta_o[(1 - x^n)/(1 - x)] \ .$$

where

$$x = e^{-(\Delta t/\tau - j\lambda\Delta t)} \ . \tag{5.92}$$

Substituting x back in θ_n gives

$$\theta_n = \theta_o\left[\frac{1 - e^{-n(\Delta t/\tau - j\lambda\Delta t)}}{1 - e^{-(\Delta t/\tau - j\lambda\Delta t)}}\right] \tag{5.93}$$

whose magnitude is

$$|\theta_n| = \theta_o\left[\frac{1 + e^{-2n\Delta t/\tau} - 2e^{-n\Delta t/\tau}\cos(n\lambda\Delta t)}{1 + e^{-2\Delta t/\tau} - 2e^{-\Delta t/\tau}\cos(\lambda\Delta t)}\right]^{1/2} \ . \tag{5.94}$$

As

$$\tau \to \infty; \ \ |\theta_n| \to \theta_o\left|\frac{\sin(n\lambda\Delta t/2)}{\sin(\lambda\Delta t/2)}\right| \ . \tag{5.95}$$

Pulsing at resonance with nutation

$$\lambda\Delta t \to 2m\pi; \ \ |\theta_n| \to \theta_o\left[\frac{1 - e^{-n\Delta t/\tau}}{1 - e^{-\Delta t/\tau}}\right] \to n\theta_o, \text{ as } \tau \to \infty \ . \tag{5.96}$$

One of the most common instances of nutation buildup is spin period attitude trim. However, with insertion of the proper Δt the relation is valid for any pulse rate greater or less than spin rate provided impulses are uniformly spaced in the vector basis with nutation frequency λ, e.g., uniform stepping of a platform or rotor mounted instrument. The reference* gives a catalog of impulse magnitudes for various types of instrument stepping disturbances. A unique example is spin rate pulsing of a rotor fixed thruster. This is a pulse train fixed both inertially and in the rotor, yielding $\cos n\lambda\Delta t = \cos 2n\pi\sigma$ considered in an inertial basis, or $\cos n\lambda\Delta t = \cos[2n\pi(\sigma - 1)] = \cos 2n\pi\sigma$ when considered in the rotor basis.

An additional solution given by (5.93) is as follows. Consider a rotor mounted thruster spinning at rate ω_s and impulsively fired at intervals Δt. Then the attitude perturbation after n pulses is given by (5.93) with $\lambda = \omega_s$ and $\tau \to \infty$. We have previously noted that nutation for this case is given by (5.93) with $\lambda = (\sigma - 1)\omega_s$.

IDC 4091.2/202 (HS331-3706), "Nutation Induced by Periodic Impulsive Disturbance of a Spin Stabilized Spacecraft Having a Passive Nutation Damper," L. H. Grasshoff, November 10, 1972.

Asymmetric Pulses

Here we consider pulses with a fixed orientation in some basis but alternating in sign. Then

$$\theta_1 = \theta_o$$

$$\theta_2 = \theta_1 e^{-(\Delta t/\tau - j\lambda\Delta t)} + \theta_o e^{j\pi}$$

$$\theta_n = \theta_{n-1} e^{-(\Delta t/\tau - j\lambda\Delta t)} + \theta_o e^{j(n-1)\pi} \tag{5.97}$$

$$= \theta_o e^{j(n-1)\pi} \sum_{i=1}^{n} e^{-(i-1)(\Delta t/\tau - j\lambda\Delta t + j\pi)}$$

$$= \theta_o e^{j(n-1)\pi} \sum_{i=1}^{n} x^{(i-1)} = \theta_o e^{j(n-1)\pi}[(1 - x^n)/(1 - x)] \, .$$

$$= \theta_o e^{j(n-1)\pi}\left[\frac{1 - e^{-n(\Delta t/\tau - j\lambda\Delta t + j\pi)}}{1 - e^{-(\Delta t/\tau - j\lambda\Delta t + j\pi)}}\right].$$

This gives nutation magnitude after the nth pulse of

$$|\theta_n| = \theta_o\left[\frac{1 + e^{-2n\Delta t/\tau} - 2e^{-n\Delta t/\tau}\cos[n(\lambda\Delta t - \pi)]}{1 + e^{-2\Delta t/\tau} - 2e^{-\Delta t/\tau}\cos(\lambda\Delta t - \pi)}\right]^{1/2} . \tag{5.98}$$

In this case, as

$$\tau \to \infty; \;\; |\theta_n| \to \theta_o\left|\frac{\sin[n(\lambda\Delta t - \pi)/2]}{\sin[(\lambda\Delta t - \pi)/2]}\right| . \tag{5.99}$$

while at resonance

$$\lambda\Delta t \to (2m - 1)\pi; \;\; |\theta_n| \to \theta_o\left[\frac{1 - e^{-n\Delta t/\tau}}{1 - e^{-\Delta t/\tau}}\right] \to n\theta_o, \text{ as } \tau \to \infty . \tag{5.100}$$

A unique application of this is rotor fixed thruster pulsing at twice spin rate such that $\Delta t = \pi/\omega_s$. Considered in a despun basis $\lambda = \lambda_o = \sigma\omega_s$, and the pulses alternate so the nutation is given by the result of this section. Alternately, considered in the rotor basis $\lambda = \lambda_s = (\sigma - 1)\omega_s$ and the result of the previous section applies and predicts the same result.

6.0 Miscellaneous Dual-Spin Dynamics Phenomena

6.1 Nutation Resonance

Nutation singular points are discussed in Appendix B, however two rather unique and well known singular points with somewhat more analytical detail in this and the following section. Consider a dual-spin spacecraft with rotor and platform spinning respectively at ω_s and ω_p. The approximate platform rates derived from small angle linearized equations are given by Equation 5.28. By a proper choice of bases $\mathbf{e}_p$, we can write the platform rate as

$$\boldsymbol{\omega}_p = \mathbf{e}_p^T[\omega_1,\, 0,\, \omega_p]^T \ , \tag{6.1}$$

where

$$\omega_1 = -\,\omega_p^2\sqrt{I_{13}^2 + I_{23}^2}\,/[\lambda_2 I_{22}] = -\,\omega_p\sqrt{I_{13}^2 + I_{23}^2}\,/[I_{33}^s\omega_s/\omega_p + I_{33}^p - I_{11}] \ . \tag{6.2}$$

Here ω_1 will be along a platform axis normal to the plane of the product $\sqrt{I_{13}^2 + I_{23}^2}$. The corresponding cone angle between the spin axis and the angular momentum vector is from (5.30 and 31).

$$\tan\theta_c = \omega_1/\omega_p = \sqrt{I_{13}^2 + I_{23}^2}\,/[I_{33}^s\omega_s/\omega_p + I_{33}^p - I_{11}] \ . \tag{6.3}$$

To get this solution, recall that we have assumed a symmetric balanced rotor. Also, nutation has been ignored; however, for small θ_c the system is linear and this "free" solution can be added by superposition. If we now admit a rotor transverse asymmetry, $I_{11}^s \neq I_{22}^s$ (or equivalently, $I_{12}^s \neq 0$), I_{11} may be expressed with an appropriate choice of basis as

$$I_{11} = I_{11}^p + \{[I_{22}^s + I_{11}^s]/2\} - \{[I_{22}^s - I_{11}^s]/2\}\cos 2\omega_r t = I_T + [\Delta I_s/2]\cos 2\omega_r t \ . \tag{6.4}$$

Hence, the rotor asymmetry is seen to cause the platform transverse rate vector and the cone angle to vary at twice relative rate. Both ω_1 and θ_c have approximately constant values to which are added a smaller component varying at $2\omega_r$. Numerous dynamic simulations have shown that during a platform spinup or down when this frequency and the platform nutation frequency become approximately equal, the platform coning motion due to rotor asymmetry induces a rapid and substantial buildup of cone angle (nutation). The phenomenon has become recognized as "nutation resonance."

An approximate relationship between ω_s and ω_p at resonance can be obtained. We write momentum as

$$\mathbf{H} = \mathbf{e}_p^T[H_1,\, 0,\, H_3]^T = \mathbf{e}_p^T H[\sin\theta_c,\, 0,\, \cos\theta_c]^T \tag{6.5}$$

where $H = |\mathbf{H}|$. Then equating twice the relative rate to platform nutation frequency

$$2\omega_r = 2(\omega_s - \omega_p) = \lambda_p = \lambda_o - \omega_p = H/I_T - \omega_p = \frac{H_3}{[I_T\cos\theta_c]} - \omega_p \tag{6.6}$$

$$= \frac{I_{33}^s\omega_s + I_{33}^p\omega_p}{I_T\cos\theta_c} - \omega_p = \sigma_s\omega_s/\cos\theta_c - \omega_p \ ,$$

and rearranging

$$\omega_p = \left[\frac{2\cos\theta_c - I_{33}^s/I_T}{\cos\theta_c + I_{33}^p/I_T}\right]\omega_s \ . \tag{6.7}$$

For small θ_c, the approximate

$$\omega_p \approx \left[\frac{2 - I_{33}^s/I_T}{1 + I_{33}^p/I_T}\right]\omega_s \ . \tag{6.8}$$

Denoting the rotor rate with the platform despun as ω_{so}, and considering a platform spinup which approximately conserves momentum in the spin axis

$$\omega_s \approx \omega_{so} - I_{33}^p\omega_p/I_{33}^s \ . \tag{6.9}$$

This leads to the approximate relation at resonance of

$$\omega_p \approx \left[\frac{2 - I_{33}^s/I_T}{1 + 2I_{33}^p/I_{33}^s} \right] \omega_{so} = \left[\frac{2 - \sigma_o}{1 + 2I_{33}^p/I_{33}^s} \right] \omega_{so} \ . \tag{6.10}$$

In such a spinup, the final platform and rotor rates are equal, and closely

$$\omega_f = I_{33}^s \omega_{so} / [I_{33}^s + I_{33}^p] \ . \tag{6.11}$$

Requiring $\omega_f > \omega_p$ where ω_p is the resonant value above yields

$$I_{33}^p > \left[\frac{(1 - \sigma_o)}{\sigma_o} \right] I_{33}^s \ . \tag{6.12}$$

If $\sigma > 1$ the right side is negative, indicating that for this case resonance is always encountered in a spinup. If $\sigma_f = [I_{33}^s + I_{33}^p]/I_T < 1$, the spinup will eventually diverge to a flat spin with θ_c approaching 90^o, in which case numerous approximations above do not hold. Extensive simulations of present day vehicles show that in this case resonance usually occurs near the end of flat spin divergence and is rather benign. For the case $\sigma_o < 1 < \sigma_f$, the above approximations have been found reasonably good when the platform imbalance is small enough to keep θ_c small. However, in such a case, a platform spinup results in a short period of spin about an intermediate axis of inertia during which θ_c diverges as an overdamped exponential (closely). To some degree, this latter effect may be inseparable from nutation resonance.

6.2 Nutation Phase Lock*

If the despun platform on a dual-spin vehicle having $\sigma < 1$ (gyrostat) is permitted to spinup, it will at some point develop a spin rate ω_p equal to inertial nutation frequency λ_o. When these two rates are equal, the centrifugal torques due to platform dynamic imbalance rotate at the same rate as the vehicle transverse rate vector. If insufficient despin torque is available to further spin up of the platform, the above torque and rate vector become phase locked such that the dynamic torque adds momentum in the transverse plane as despin torque (friction et. al.) removes it from the spin axis. In this phase locked condition, transverse rate (nutation angle) grows linearly with time producing an eventual divergence to flat spin. Equations for transverse rate and nutation angle in the phase locked condition are derived below.

Consider a dual-spin vehicle configuration with symmetric balanced rotor and $I^p_{12} = 0$. Then Equation 2.34 applies and, truncating small rate inertia products, it reduces to

$$\mathbf{e}_p^T \begin{bmatrix} I_T\dot{\omega}_1 + \lambda_p I_T \omega_2 \\ I_T\dot{\omega}_2 - \lambda_p I_T \omega_1 \\ I^p_{33}\dot{\omega}_3 \end{bmatrix} = \mathbf{e}_p^T \begin{bmatrix} -I_{23}\omega_p^2 \\ I_{13}\omega_p^2 \\ T_3 \end{bmatrix} \tag{6.13}$$

assuming no external torques. We wish to transform to an "approximately inertial" despun basis $\mathbf{e}_o$ by removing the platform spin ω_p, i.e.

$$\mathbf{e}_p = \begin{bmatrix} \cos\phi_p & \sin\phi_p & 0 \\ -\sin\phi_p & \cos\phi_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_o \ . \tag{6.14}$$

Then the platform rate transforms as

$$\begin{aligned} \boldsymbol{\omega}_p &= \mathbf{e}_p^T[\omega_1,\ \omega_2,\ \omega_3 + \omega_p]^T \\ &= \mathbf{e}_o^T[\omega_1\cos\phi_p - \omega_2\sin\phi_p,\ \omega_2\cos\phi_p + \omega_1\sin\phi_p,\ \omega_3 + \omega_p]^T = \mathbf{e}_o^T[\omega_x,\ \omega_y,\ \omega_3 + \omega_p]^T\ , \end{aligned} \tag{6.15}$$

and the angular accelerations similarly.

Transforming the torque equations and substituting the rate expressed in $\mathbf{e}_o$ gives,

$$\mathbf{e}_o^T \begin{bmatrix} I_T\dot{\omega}_x + \lambda_o I_T \omega_y \\ I_T\dot{\omega}_y - \lambda_o I_T \omega_x \\ I^p_{33}\dot{\omega}_3 \end{bmatrix} = \mathbf{e}_o^T \begin{bmatrix} -\omega_p^2\sqrt{I_{13}^2 + I_{23}^2}\cos(\phi_p - \gamma) \\ -\omega_p^2\sqrt{I_{13}^2 + I_{23}^2}\sin(\phi_p - \gamma) \\ T_3 \end{bmatrix}, \tag{6.16}$$

with

$$\sin\gamma = I_{23}/\sqrt{I_{13}^2 + I_{23}^2} \tag{6.17a}$$

$$\cos\gamma = I_{13}/\sqrt{I_{13}^2 + I_{23}^2}\ . \tag{6.17b}$$

Setting $\phi_p = \omega_p t$ and applying sufficient perseverance, the solution is

$$\omega_x(t) = \omega_x(0)\cos\lambda_o t - \omega_y(0)\sin\lambda_o t - \frac{\omega_p^2\sin\gamma\sqrt{I_{13}^2 + I_{23}^2}}{(\lambda_o - \omega_p)I_T}\Big[-\{\cos\lambda_o t - \cos\omega_p t\} + (1/\tan\gamma)\{\sin\lambda_o t - \sin\omega_p t\}\Big] \tag{6.18a}$$

* A more precise derivation than given here yields the phase lock platform rate as $\omega_p = (H/I_T)\{1 - (I_{23}/I_T)^{2/3} + (I^s_{22} - I^s_{11})/2I_T\}$.

$$\omega_y(t) = \omega_y(0)\cos\lambda_o t + \omega_x(0)\sin\lambda_o t - \frac{\omega_p^2 \cos\gamma\sqrt{I_{13}^2 + I_{23}^2}}{(\lambda_o - \omega_p)I_T}\left[-\{\cos\lambda_o t - \cos\omega_p t\} - \tan\gamma\{\sin\lambda_o t - \sin\omega_p t\}\right]. \quad (6.18b)$$

Combining the sinusoidal sum terms as products at sum and difference frequencies,

$$\omega_x(t) = \omega_x(0)\cos\lambda_o t - \omega_y(0)\sin\lambda_o t - \frac{\omega_p^2\sqrt{I_{13}^2 + I_{23}^2}}{I_T}\left[\frac{\sin[(\lambda_o - \omega_p)t/2]}{[(\lambda_o - \omega_p)/2]}\right]\cos[(\lambda_o + \omega_p)t/2 - \gamma] \quad (6.19a)$$

$$\omega_y(t) = \omega_y(0)\cos\lambda_o t + \omega_x(0)\sin\lambda_o t - \frac{\omega_p^2\sqrt{I_{13}^2 + I_{23}^2}}{I_T}\left[\frac{\sin[(\lambda_o - \omega_p)t/2]}{[(\lambda_o - \omega_p)/2]}\right]\sin[(\lambda_o + \omega_p)t/2 - \gamma]\ . \quad (6.19b)$$

Setting the initial conditions to zero, the transverse rate magnitude is

$$\omega_T(t) = \frac{t\omega_p^2\sqrt{I_{13}^2 + I_{23}^2}}{I_T}\left[\frac{\sin(\lambda_o - \omega_p)t/2}{(\lambda_o - \omega_p)t/2}\right]. \quad (6.20)$$

Thus, in the limit at resonance $\lambda_o \to \omega_p$, $\omega_T(t)$ grows linearly with time as

$$\omega_T(t) \to \frac{t\lambda_o^2\sqrt{I_{13}^2 + I_{23}^2}}{I_T}\ . \quad (6.21)$$

Integrating the forced rate response in the limiting case as $\lambda_o \to \omega_p$ yields

$$\theta_x(t) = \frac{-\sqrt{I_{13}^2 + I_{23}^2}}{I_T}\left[\cos(\lambda_o t - \gamma) + \lambda_o t\sin(\lambda_o t - \gamma) - \cos\gamma\right] \quad (6.22a)$$

$$= -\left[I_{13}\cos\lambda_o t + I_{23}\sin\lambda_o t + \lambda_o t\{I_{13}\sin\lambda_o t - I_{23}\cos\lambda_o t\} - I_{13}\right]/I_T$$

$$\theta_y(t) = \frac{-\sqrt{I_{13}^2 + I_{23}^2}}{I_T}\left[\sin(\lambda_o t - \gamma) - \lambda_o t\cos(\lambda_o t - \gamma) + \sin\gamma\right] \quad (6.22b)$$

$$= -\left[I_{13}\sin\lambda_o t - I_{23}\cos\lambda_o t - \lambda_o t\{I_{13}\cos\lambda_o t + I_{23}\sin\lambda_o t\} + I_{23}\right]/I_T\ .$$

From these equations the spin axis divergence path is sketched as Figure 6.1 below.

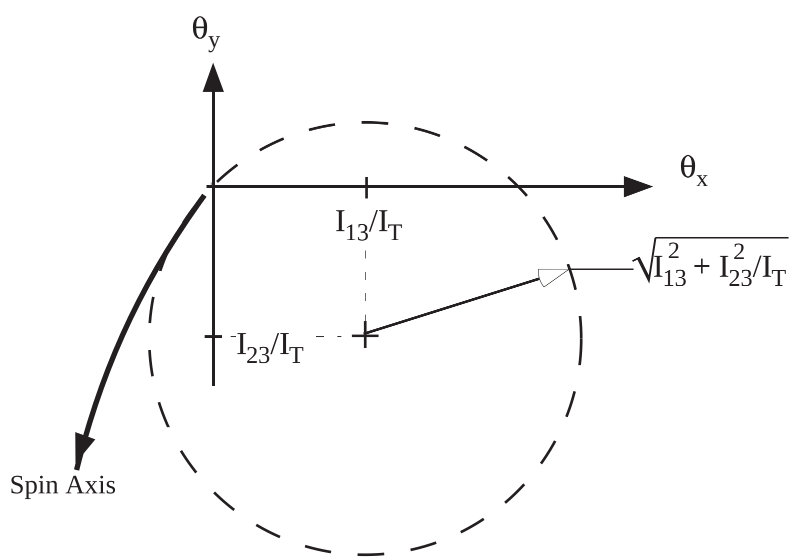


Figure 6.1 Trajectory of Spin Axis Divergence on Inertial Plane in Phase Locked State.

6.3 Nutation Spinup

An approach sometimes employed to increase the spin momentum of a spinning spacecraft is to induce transverse axis momentum (nutation) with a thruster and then transfer this momentum to the spin axis via a momentum conserving nutation damper. If ω_s is the initial spin rate with no nutation, and nutation angle θ is induced with thrusters, the initial and final momenta are

$$H_i = I_{33}^s \omega_s \tag{6.23}$$

$$H_f = I_{33}^s \omega_s / \cos\theta\ . \tag{6.24}$$

Hence,

$$\Delta H = H_f - H_i = I_{33}^s \omega_s [1/\cos\theta - 1] = I_{33}^s \Delta\omega_s \tag{6.25}$$

where $\Delta\omega_s$ is the change in spin speed that will result when the induced nutation is transferred to the spin axis. For small angles, the latter approximates to

$$\Delta H \approx I_{33}^s \omega_s \theta^2/2 \approx I_{33}^s \Delta\omega_s\ . \tag{6.26}$$

An efficient approach to induce nutation is to apply thruster torque T for half rotor nutation periods. Let

$$|\lambda_s| = |\sigma - 1|\omega_s \tag{6.27}$$

be the rotor nutation frequency. Then the torque pulses are of duration $\Delta t = \pi/|\lambda_s|$ and Appendix C shows that when properly phased, each pulse will induce nutation

$$\theta = 2\rho = 2T/H|\lambda_s| = 2T/[I_{33}^s|\sigma - 1|\omega_s^2]\ . \tag{6.28}$$

Substituting this in the small angle approximation of ΔH, the spin speed change per pulse is

$$\Delta\omega_s = [T/(I_{33}^s \lambda_s)]^2 [2/\omega_s] = [T/(I_{33}^s|\sigma - 1|)]^2 [2/\omega_s^3]\ . \tag{6.29}$$

If the nutation is induced by a thruster with transverse torque $r_n f_n$, spinup fuel sensitivity can be written

$$S_n = \frac{\Delta w}{\Delta\omega_s} = \left[\frac{f_n \Delta t}{I_{sp}}\right]\frac{1}{\Delta\omega_s} = \left[\frac{T\pi}{r_n|\lambda_s|I_{sp}}\right]\frac{1}{\Delta\omega_s} = \left[\frac{\pi/2}{r_n^2 f_n I_{sp}}\right] {I_{33}^s}^2 \omega_s |\lambda_s|\ .$$

For a spin thruster with spin torque $r_s f_s$, the spin increment is $\Delta\omega = [r_s f_s \Delta t]/I_{33}^s$, while fuel consumption sensitivity is

$$S_s = \frac{\Delta w}{\Delta\omega_s} = \left[\frac{f_s \Delta t}{I_{sp}}\right]\frac{1}{\Delta\omega_s} = \frac{I_{33}^s}{r_s I_{sp}}$$

Then nutation spin up is more efficient when

$$\frac{S_s}{S_n} = \left[\frac{I_{33}^s}{r_s I_{sp}}\right]\left[\frac{r_n^2 f_n I_{sp}}{(\pi/2){I_{33}^s}^2 \omega_s|\lambda_s|}\right] = \left[\frac{1}{r_s}\right]\left[\frac{r_n^2 f_n}{(\pi/2) I_{33}^s \omega_s |\lambda_s|}\right] > 1\ .$$

6.4 Allspun Recovery Static Motor Torque Bounds and Rocking Frequency

Let $\mathbf{e}_i$ be an inertial basis with 3-axis along the momentum vector as depicted below on Figure 6.2. Assuming no external torques, this vector is inertially fixed. Denote the spacecraft inertial spin phase about the momentum vector by ψ and the cone angle by

$$\theta_c = \mathrm{Tan}^{-1}\left[\frac{\sqrt{H_1^2 + H_2^2}}{H_3}\right] \tag{6.30}$$

where

$$\mathbf{H} = \mathbf{e}_p^T H = \mathbf{e}_p^T [H_1, H_2, H_3]^T \tag{6.31}$$

and $\mathbf{e}_p$ is a platform fixed basis with 3-axis along the bearing axis. Define a third basis $\mathbf{e}_b$ with 3-axis along the bearing axis and 1-axis along the transverse momentum vector $\mathbf{H}^T = \mathbf{e}_p^T[H_1, H_2, 0]^T$. Finally, denoting the displacement of $\mathbf{e}_p$ from $\mathbf{e}_b$ as β_p, the bases are related as

$$\mathbf{e}_b = C_1 \mathbf{e}_i = \begin{bmatrix} \cos\theta_c & 0 & -\sin\theta_c \\ 0 & 1 & 0 \\ \sin\theta_c & 0 & \cos\theta_c \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_i \tag{6.32}$$

$$\mathbf{e}_p = C_2 \mathbf{e}_b = \begin{bmatrix} \cos\beta_p & \sin\beta_p & 0 \\ -\sin\beta_p & \cos\beta_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_b \tag{6.33}$$

with

$$\beta_p = \mathrm{Tan}^{-1}[H_2/H_1]\ . \tag{6.34}$$

The inertial angular rate of $\mathbf{e}_b$ is then

$$\boldsymbol{\omega}_b = \mathbf{e}_b^T[0, \dot\theta_c, 0]^T + \mathbf{e}_b^T C_1 [0, 0, \dot\psi]^T = \mathbf{e}_b^T[-\dot\psi\sin\theta_c, \dot\theta_c, \dot\psi\cos\theta_c]^T = \mathbf{e}_b^T \omega_b \tag{6.35}$$

and the platform inertial rate is

$$\boldsymbol{\omega}_p = \boldsymbol{\omega}_b + \mathbf{e}_p^T[0, 0, \dot\beta_p]^T = \mathbf{e}_p^T \begin{bmatrix} -\dot\psi\sin\theta_c\cos\beta_p + \dot\theta_c\sin\beta_p \\ \dot\psi\sin\theta_c\sin\beta_p + \dot\theta_c\cos\beta_p \\ \dot\psi\cos\theta_c + \dot\beta_p \end{bmatrix} = \mathbf{e}_p^T\omega_p = \mathbf{e}_p^T \begin{bmatrix} \omega_{p1} \\ \omega_{p2} \\ \omega_{p3} \end{bmatrix} . \tag{3.36}$$

The dynamic torques exerted by the platform along the bearing axis (3-axis of $\mathbf{e}_p$ and $\mathbf{e}_b$) are obtained by differentiating $\boldsymbol{\omega}_p$ and substituting in Eq. 2.16. The resultant torque is

$$T_3 = -I_{13}^p\dot\omega_{p1} - I_{23}^p\dot\omega_{p2} + I_{33}^p\dot\omega_{p3} + [I_{22}^p - I_{11}^p]\omega_{p1}\omega_{p2} - I_{23}^p\omega_{p1}\omega_{p3} + I_{13}^p\omega_{p2}\omega_{p3} + I_{12}^p[\omega_{p2}^2 - \omega_{p1}^2]\ . \tag{6.37}$$

Substituting ω_p, $\dot\omega_p$ in terms of ψ, θ_c, and β_p, and taking all derivatives closely zero the approximate static torque bound reduces to

$$T_3 = (\dot\psi^2/2)[-\Delta\bar{I}_p \sin^2\theta_c \sin\{2(\beta_p + \gamma_1)\} + \sqrt{{I_{13}^p}^2 + {I_{23}^p}^2}\,\sin 2\theta_c \sin(\beta_p + \gamma_2)]\ , \tag{6.38}$$

where

$$\Delta\bar{I}_p = |\bar{I}_{22}^p - \bar{I}_{11}^p| = \sqrt{(I_{22}^p - I_{11}^p)^2 + (2I_{12}^p)^2} \tag{6.39}$$

$$\gamma_1 = (1/2)\mathrm{Tan}^{-1}[2I_{12}^p/(I_{22}^p - I_{11}^p)] \tag{6.40}$$

$$\gamma_2 = \mathrm{Tan}^{-1}[I_{23}^p/I_{13}^p] \tag{6.41}$$

and

$$\dot\psi = H/I_m \tag{6.42}$$

with I_m the maximum principal axis of the all-spun vehicle with the rotor to platform rotation adjusted to maximize I_m. $\Delta\bar{I}_p$ is the platform transverse inertia difference with $\mathbf{e}_p$ chosen such that $I_{12}^p = 0$. Thus, the maximum $T_3 > 0$ over β_p ε $(0, 2\pi)$ is the static torque limit which must not be exceeded to avoid platform spinup during flat spin recovery of a prolate ($\sigma < 1$) dual-spin vehicle. Similarly this bound must be exceeded to despin the platform on an all-spun oblate ($\sigma > 1$) dual-spin vehicle.

The torque limit T_3 derived above is the static torque required to hold the two bodies in a fixed relative position β_p as a function of that position. For some β_p, in the equilibrium trap state, $T_3 = 0$ and at some other point it is maximized. The latter torque is what must be overcome in a static sense to initiate relative spin between the two bodies.

If $I^p_{13} = I^p_{23} = 0$, the inertia matrix may be expressed in a body coordinate system rotated from $\mathbf{e}_p$ by the angle γ_1 such that $\bar{I}^p_{12} = 0$, and the torque bound reduces to

$$T_3 = (\dot{\psi}^2/2)\Delta\bar{I}_p \sin^2\theta_c \sin 2\beta_p \ . \tag{6.43}$$

The parallel bound for the rotor is derived in the same manner, and may be obtained from T_3 by inserting rotor inertias.

If sufficient motor torque does not exist to overcome the maximum static torque given by T_3 above the allspun trap state can sometimes be overcome by rocking the two dual spin bodies with respect to each other. A natural frequency is established by the vehicle momentum state and mass properties for small motions about the allspun equilibrium state. By linearizing the equation for T_3 about the equilibrium β_p and denoting perturbations by β_p, we get

$$T_3 = I^p_{33}\ddot{\beta}_p + (\dot{\psi}^2/2)[2\Delta\bar{I}_p \sin^2\theta_c - \sqrt{{I^p_{13}}^2 + {I^p_{23}}^2}\sin 2\theta_c]\beta_p = I^p_{33}[\ddot{\beta}_p + \Omega^2\beta_p] \ . \tag{6.44}$$

For flat spin $\theta_c = 90^o$ and the natural frequency is

$$\Omega^2 \approx \dot{\psi}^2[\Delta\bar{I}_p/I^p_{33}] \ . \tag{6.45}$$

By pulsing the torque motor at this frequency a recovery can be initiated with much less torque than that given by the static bound. More generally one can solve for the steady state allspun cone angle θ_c and substitute in the restraining torque equation T_3. For a symmetric vehicle with $I_{13} = 0$ in the allspun state

$$\tan 2\theta_c = I_{23}/[(I_T - I_{33})/2] \tag{6.46a}$$

$$\sin 2\theta_c = -\,I_{23}/\{[(I_T - I_{33})/2]^2 + I^2_{23}\}^{1/2} \tag{6.46b}$$

$$\cos 2\theta_c = -\,[(I_T - I_{33})/2]/\{[(I_T - I_{33})/2]^2 + I^2_{23}\}^{1/2} \tag{6.46c}$$

where the inertias are for the total allspun vehicle.

If both rotor and platform are statically imbalanced the trap state becomes much more severe. The relationship of the two body mass centers and the bearing axis for this case are depicted on Figure 6.3. Some static torque maintains angle β_p between the two bodies as depicted. For a stable spinner the spin vector is approximately out of the page as indicated by the circles, while for a flat spin condition the spin vector will be in the page approximately normal to the line joining the mass centers. In either case it is clear the body mass centers tend to opposite sides of the bearing axis and some minimum torque will be required to initiate more than 90^o of relative motion.

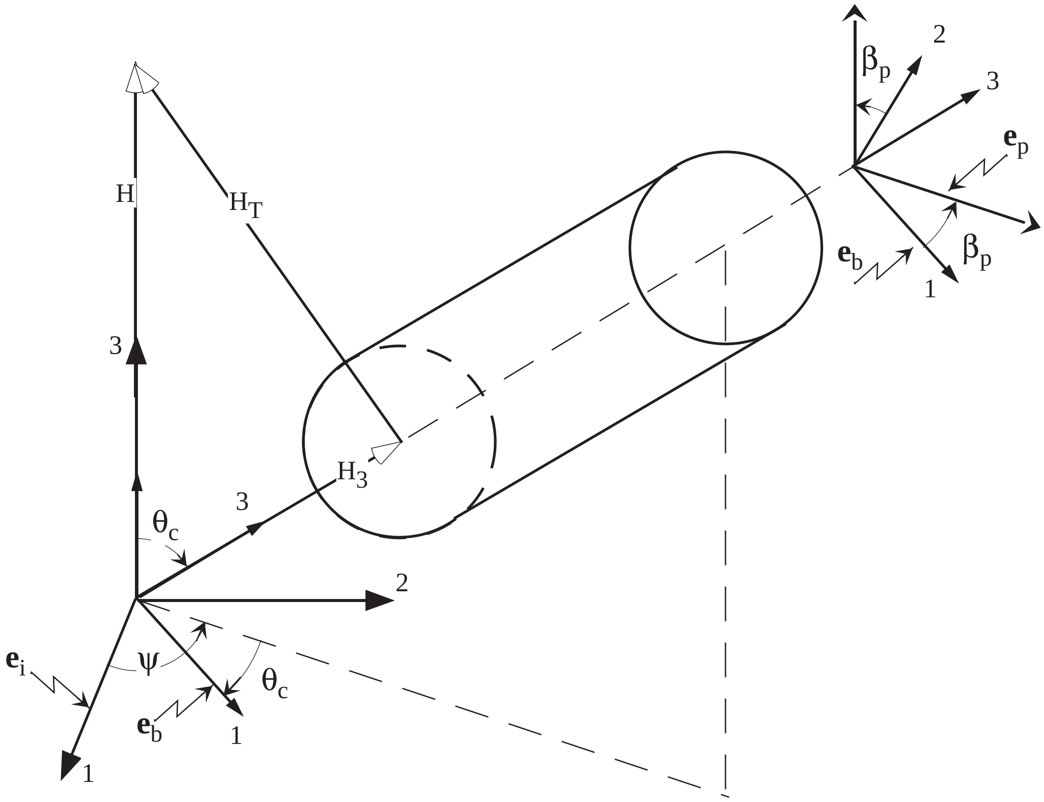


Figure 6.2 Geometry for Analysis of Dynamic Imbalance Trap State.

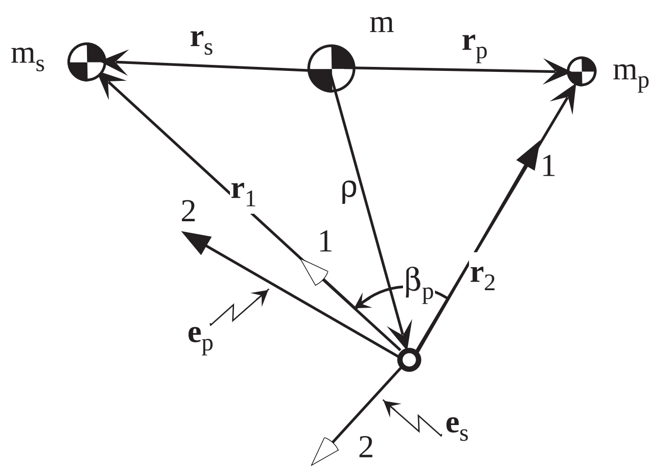


Figure 6.3 Geometry of the Static Imbalance Trap State.

For this case a somewhat different approach is taken to derive the bound. From the bearing bending restraint equation derived earlier (4.21) we have

$$\mathbf{M}_p = \mathbf{J}_p \cdot \dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p \times [\mathbf{J}_p \cdot \boldsymbol{\omega}_p] + m_p \mathbf{r}_2 \times \ddot{\mathbf{r}}_p \tag{6.47}$$

using the notation of Figure 6.3 and letting $\mathbf{J}_p$ denote inertia of the platform with respect to its own cm. Using the center of mass definition it is found that

$$\mathbf{r}_p = (m_s/m)[\mathbf{r}_2 - \mathbf{r}_1] \ . \tag{6.48}$$

Since we are interested in the static case $\dot{\omega}_p = \dot{\omega}_s = 0$, and $\omega_p = \omega_s$. Also both $\mathbf{r}_1$, $\mathbf{r}_2$ are fixed in a basis of either body while the two bodies are again at relative phase β_p. Then denote

$$\mathbf{r}_p = \mathbf{e}_p^T (m_s/m)[x_2 - x_1 \cos\beta_p, - x_1 \sin\beta_p, z_2 - z_1]^T \ , \tag{6.49}$$

while

$$\ddot{\mathbf{r}}_p \approx \boldsymbol{\omega}_p \times [\boldsymbol{\omega}_p \times \mathbf{r}_p] \ . \tag{6.50}$$

Thus, the spin axis component of $\mathbf{M}_p$ is

$$M_3 = (J^p_{22} - J^p_{11})\omega_1\omega_2 + J^p_{12}(\omega_2^2 - \omega_1^2) + J^p_{13}\omega_2\omega_3 - J^p_{23}\omega_1\omega_3 \tag{6.51}$$

$$+ m_p(m_s/m)\{x_1(z_2 - z_1)\omega_2\omega_3 + x_2(x_2 - x_1\cos\beta_p)\omega_1\omega_2 + x_1x_2\sin\beta_p(\omega_2^2 + \omega_3^2)\} \ .$$

Of course the rates here are in general very complex functions of the mass properties and presumably this will reduce to the previous bound T_3 with sufficient effort. Of significant interest is the simple dynamically balanced and symmetric inertia case where the vehicle is allspun about an axis near the bearing axis. Then the recovery torque bound simplifies to

$$M_3 = m_p(m_s/m)\omega_3^2 x_1 x_2 \sin\beta_p \ . \tag{6.52}$$

Similarly, in flat spin we reason from Figure 6.3 that the primary rate will be ω_2, and the bound is

$$M_3 = m_p(m_s/m)\omega_2^2 x_1 x_2 \sin\beta_p \ . \tag{6.53}$$

6.5 Separation Dynamics

Consider two bodys initially joined as a single rigid body that are separated by an internal force acting between them. Given mass properties, the separating force **F**(t), and initial linear and angular velocities **v**, **Ω**, we wish to calculate linear and angular velocities of the separated bodies. Figure 6.4a depicts problem geometry and defines various position and rate vectors.

Immediately following initiation of the separation when the interbody constraints are released and the internal force begins to act, body a has angular momentum

$$\begin{aligned}\mathbf{H}_a &= \int [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1 + \mathbf{u}_1] \times [\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_1 + \dot{\mathbf{x}}_1 + \dot{\mathbf{u}}_1] dm_1 \\ &= m_1 [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1] \times [\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_1 + \dot{\mathbf{x}}_1] + \int \mathbf{u}_1 \times \dot{\mathbf{u}}_1 dm_1 \qquad (6.54) \\ &= m_1 [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1] \times [\dot{\mathbf{r}}_o + \dot{\mathbf{r}}_1 + \dot{\mathbf{x}}_1] + \mathbf{J}_a \cdot \boldsymbol{\omega}_a \ ,\end{aligned}$$

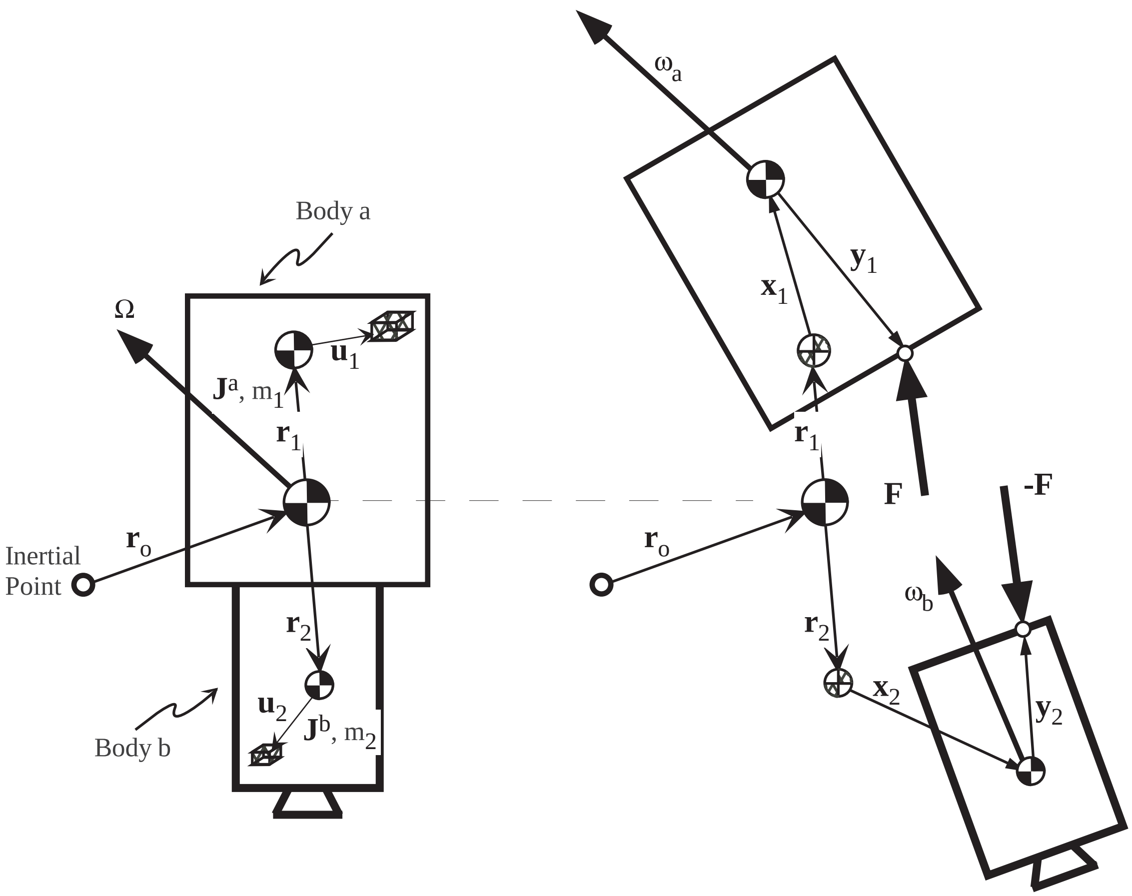


Figure 6.4a Two Body Separation Geometry.

and differentiating

$$\begin{aligned}\dot{\mathbf{H}}_a &= \int [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1 + \mathbf{u}_1] \times [\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_1 + \ddot{\mathbf{x}}_1 + \ddot{\mathbf{u}}_1] dm_1 \\ &= m_1 [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1] \times [\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_1 + \ddot{\mathbf{x}}_1] + \int \mathbf{u}_1 \times \ddot{\mathbf{u}}_1 dm_1 \qquad (6.55) \\ &= m_1 [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1] \times [\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_1 + \ddot{\mathbf{x}}_1] + \mathbf{J}_a \cdot \dot{\boldsymbol{\omega}}_a + \boldsymbol{\omega}_a \times [\mathbf{J}_a \cdot \boldsymbol{\omega}_a] \ .\end{aligned}$$

The moment on body a is

$$\mathbf{M}_a = [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1 + \mathbf{y}_1] \times \mathbf{F} = m_1 [\mathbf{r}_o + \mathbf{r}_1 + \mathbf{x}_1] \times [\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_1 + \ddot{\mathbf{x}}_1] + \mathbf{y}_1 \times \mathbf{F} \ . \qquad (6.56)$$

Equating (6.55) and (6.56), the body a moment equation becomes

$$\mathbf{J}_a \cdot \dot{\boldsymbol{\omega}}_a + \boldsymbol{\omega}_a \times [\mathbf{J}_a \cdot \boldsymbol{\omega}_a] = \mathbf{y}_1 \times \mathbf{F} \tag{6.57}$$

where $\mathbf{F}$ is given and at time zero $\boldsymbol{\omega}_a = \boldsymbol{\Omega}$ is given. Of course there may be multiple forces (torques) that sum on the right side of (6.57). Initial linear velocity is

$$\mathbf{v}_1 = \dot{\mathbf{r}}_o + \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_o + \boldsymbol{\Omega} \times \mathbf{r}_1 \ . \tag{6.58}$$

Acceleration after release is

$$\ddot{\mathbf{x}}_1 = \mathbf{F}/m_1 \tag{6.59}$$

which results in linear velocity

$$\mathbf{v}_1 + \Delta\mathbf{v}_1 = \dot{\mathbf{r}}_o + \boldsymbol{\Omega} \times \mathbf{r}_1 + \int \ddot{\mathbf{x}} dt \ . \tag{6.60}$$

The solution of (6.57) and (6.59) is the sought result for body a. An entirely parallel set of equations describe motion of body b. However, such a separation is usually executed by letting $\mathbf{F}$ be a spring between the two bodies. In general this yields $\mathbf{F}$ as a function of the position of both bodies. Since the force is internal $m_1\ddot{\mathbf{x}}_1 = -m_2\ddot{\mathbf{x}}_2$ and $\mathbf{x}_2 - \mathbf{x}_1 = -(1 + m_1/m_2)\mathbf{x}_1$. Thus, the internal spring force will be a function of the body position difference and both body rates (orientations), i.e., $\mathbf{F} = \mathbf{F}(\mathbf{x}_1, \boldsymbol{\omega}_a, \boldsymbol{\omega}_b)$. As a result, for the general case one must integrate 9 dynamic equations simultaneously to obtain $\mathbf{x}_1, \boldsymbol{\omega}_a, \boldsymbol{\omega}_b$. These equations are (6.57), (6.59), and the counterpart to (6.57) for body b.

Expanding (6.57) for a dynamically balanced body in body fixed vector basis $\mathbf{e}_a$ gives

$$\mathbf{e}_a^T \begin{bmatrix} J_{11}^a\dot{\omega}_1 + (J_{33}^a - J_{22}^a)\omega_2\omega_3 - J_{13}^a\dot{\omega}_3 + J_{23}^a\omega_3^2 \\ J_{22}^a\dot{\omega}_2 - (J_{33}^a - J_{11}^a)\omega_1\omega_3 - J_{23}^a\dot{\omega}_3 - J_{13}^a\omega_3^2 \\ J_{33}^a\dot{\omega}_3 + (J_{22}^a - J_{11}^a)\omega_1\omega_2 + J_{13}^a\omega_2\omega_3 - J_{23}^a\omega_1\omega_3 \end{bmatrix} \rightarrow \mathbf{e}_a^T \begin{bmatrix} J_{11}^a\dot{\omega}_1 + (J_{33}^a - J_{22}^a)\omega_2\omega_3 \\ J_{22}^a\dot{\omega}_2 - (J_{33}^a - J_{11}^a)\omega_1\omega_3 \\ J_{33}^a\dot{\omega}_3 + (J_{22}^a - J_{11}^a)\omega_1\omega_2 \end{bmatrix} = \mathbf{y}_1 \times \mathbf{F} \ . \tag{6.61}$$

We shall solve the simple case for a spinning body, $\Omega_3(0) = \omega_s$, where the separation force is impulsive, and where the bodies are axis symmetric, i.e., $J_{22} = J_{11} = J_T$. Denote the force and moment impulses respectively as

$$\int \mathbf{F} dt = \mathbf{F}\tau = \mathbf{e}_a^T \tau [F_1, F_2, F_3]^T \tag{6.62}$$

and

$$\int \mathbf{y}_1 \times \mathbf{F} dt = \mathbf{T}\tau = \mathbf{e}_a^T \tau [T_1, T_2, T_3]^T \ . \tag{6.63}$$

The 3-axis equation from (6.61) integrates to spin rate of the separated body a as

$$\omega_s = \Omega_3(0) + \tau T_3/J_{33}^a \ . \tag{6.64}$$

Then the body a nutation frequency becomes

$$\lambda_a = (J_{33}^a/J_T^a - 1)\omega_s = (\sigma_a - 1)\omega_s \ . \tag{6.65}$$

Integrating the transverse axis equations from (6.61) yields final body a angular rates

$$\boldsymbol{\omega}_a = \mathbf{e}_a^T \begin{bmatrix} [\tau T_1/J_T^a + \Omega_1(0)]\cos\lambda_a t - [\tau T_2/J_T^a + \Omega_2(0)]\sin\lambda_a t \\ [\tau T_2/J_T^a + \Omega_2(0)]\cos\lambda_a t - [\tau T_1/J_T^a + \Omega_1(0)]\sin\lambda_a t \\ \omega_s \end{bmatrix} . \tag{6.66}$$

Simultaneously the velocity of body a mass center becomes

$$\mathbf{v}_1 + \Delta\mathbf{v}_1 = \dot{\mathbf{r}}_o + \boldsymbol{\Omega} \times \mathbf{r}_1 + (\tau/m_1)\mathbf{F} \tag{6.67}$$

and solutions similar to (6.66) and (6.67) hold for the opposite body. The velocity of a point on body a located at $\mathbf{y}_a$ with respect to its cm has velocity

$$\mathbf{v}_a = \mathbf{v}_1 + \Delta\mathbf{v}_1 + \boldsymbol{\omega}_a \times \mathbf{y}_a = \dot{\mathbf{r}}_o + \boldsymbol{\Omega} \times \mathbf{r}_1 + (\tau/m_1)\mathbf{F} + \boldsymbol{\omega}_a \times \mathbf{y}_a \ , \tag{6.68}$$

and the position change of this point

$$\Delta \mathbf{y}_a(t) = \int_0^t \mathbf{v}_a dt \,. \tag{6.69}$$

Observe that if **F** imparts no torque the angular rates remain unaltered by separation. However, the nutation angle excursion may change due to a different nutation frequency for the separated body. Letting σ and θ represent inertia ratio and nutation angle for the initial stack and σ_a, θ_a similarly for separated body a, the relation

$$\tan\theta_a/\tan\theta = \sigma/\sigma_a \approx \theta_a/\theta \,, \tag{6.70}$$

the latter for small angles, holds for a torque free separation. It may also be shown that the attitude of the separated body shifts by an amount equal to the nutation change in a random direction from the attitude of the original body.

Separation Clearance Reduction due to One Body Dynamic Imbalance

Here we consider the first-order effect of a dynamic imbalance J^a_{23} on body a. Assume a spinning axial separation and that the launch vehicle, body b, has perfect control imparting a perfect spin $\omega_3 = \omega_s$ about the 3-axis of the geometric coordinate basis. Note another case that we do not solve at present is when the launch vehicle exerts no control. To accomplish the former the launch vehicle will have to apply a constant body fixed torque $T_1 = J^a_{23}\omega_s^2$, see Eq. 6.61, which will force the momentum vector to cone in space. The coning angle can be found as $\theta^i_h = J^a_{23}/[J^a_{33} + J^b_{33}]$ and at the instant of release, body a has pure spin about its 3-axis and momentum vector at cone angle $\theta_h = J^a_{23}/J^a_{33}$. Hence the principal axis is displaced from the geometric axis by $\theta_w = J^a_{23}/[J^a_{33} - J^a_{22}]$, and from the momentum vector, which is fixed in inertial space after separation, by initial nutation angle

$$\theta_n = \theta_w \pm \theta_h = J^a_{23}/[J^a_{33} - J^a_{22}] \pm J^a_{23}/J^a_{33} = \{J^a_{23}/[J^a_{33} - J^a_{22}]\}\left[\frac{\sigma_2 \pm (\sigma_2 - 1)}{\sigma_2}\right] = \begin{cases} \theta_w/\sigma_2 \,; & \sigma_2 > 1 \\ \theta_w(2\sigma_2 - 1)/\sigma_2 \,; & \sigma_2 < 1 \end{cases} . \tag{6.71}$$

At the instant of release body a will have angular rate purely about spin and a step torque $-J^a_{23}\omega_s^2$ as a result of removal of the b body canceling control torque. Hence, using 5.67, 68

$$\boldsymbol{\omega}_a = -\mathbf{e}_a^T \begin{bmatrix} \theta_w\omega_s \sin\lambda_s t \\ \theta_w\omega_s[1 - \cos\lambda_s t] \\ \omega_s \end{bmatrix} = -\mathbf{e}_i^T \begin{bmatrix} \theta_n\lambda_o \sin\lambda_o t - \theta_w\omega_s \sin\omega_s t \\ -\theta_n\lambda_o \cos\lambda_o t + \theta_w\omega_s \cos\omega_s t \\ \omega_s \end{bmatrix} . \tag{6.72}$$

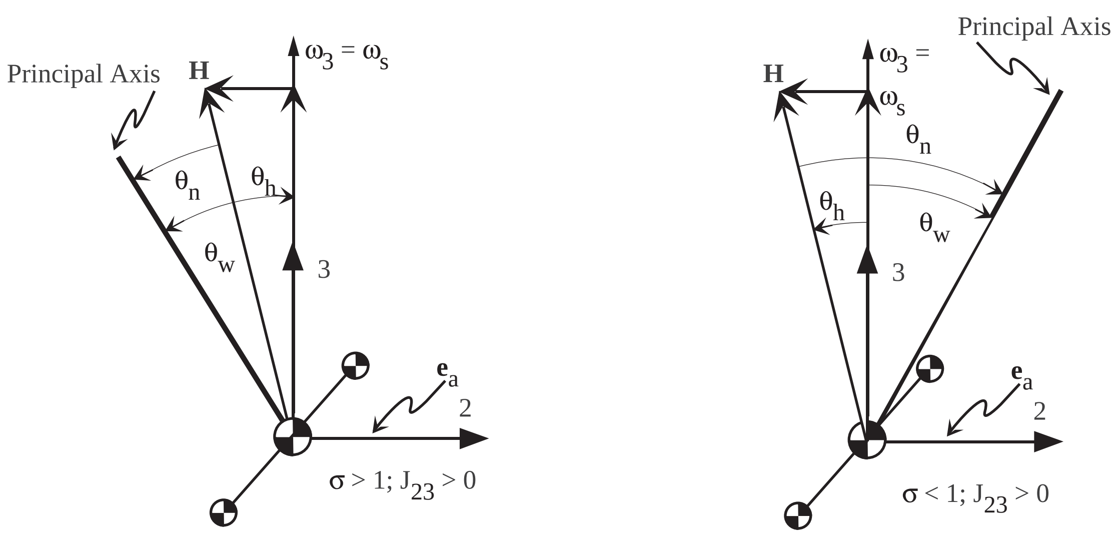


Figure 6.4b Spin and Momentum Vector Geometry at Separation.

Consider the velocity of a point on body a at

$$\mathbf{r} = \mathbf{e}_a^T[r_1, r_2, r_3]^T = \mathbf{e}_i^T[r_1 \cos \omega_s t - r_2 \sin \omega_s t,\ r_1 \sin \omega_s t + r_2 \cos \omega_s t,\ r_3] \tag{6.73}$$

which is

$$\Delta\mathbf{v}_1 = \boldsymbol{\omega}_a \times \mathbf{r} = \mathbf{e}_a^T r_3 \begin{bmatrix} \theta_w \omega_s - \theta_n \lambda_o \cos \lambda_s t \\ -\theta_n \lambda_o \sin \lambda_s t \\ 0 \end{bmatrix} + \mathbf{e}_a^T \begin{bmatrix} -r_2 \omega_s \\ r_1 \omega_s \\ 0 \end{bmatrix} = \mathbf{e}_a^T r_3 \theta_w \omega_s \begin{bmatrix} 1 - \cos \lambda_s t \\ -\sin \lambda_s t \\ 0 \end{bmatrix} + \mathbf{e}_a^T \begin{bmatrix} -r_2 \omega_s \\ r_1 \omega_s \\ 0 \end{bmatrix} \tag{6.74}$$

$$= -\,\mathbf{e}_i^T \begin{bmatrix} \theta_n \lambda_o \sin \lambda_o t - \theta_w \omega_s \sin \omega_s t \\ -\theta_n \lambda_o \cos \lambda_o t + \theta_w \omega_s \cos \omega_s t \\ \omega_s \end{bmatrix} \times \mathbf{e}_i^T \begin{bmatrix} r_1 \cos \omega_s t - r_2 \sin \omega_s t \\ r_1 \sin \omega_s t + r_2 \cos \omega_s t \\ r_3 \end{bmatrix}$$

$$= \mathbf{e}_i^T r_3 \begin{bmatrix} \theta_n \lambda_o \cos \lambda_o t - \theta_w \omega_s \cos \omega_s t \\ \theta_n \lambda_o \sin \lambda_o t - \theta_w \omega_s \sin \omega_s t \\ v_3 \end{bmatrix} + \mathbf{e}_i^T \omega_s \begin{bmatrix} -\,r_1 \sin \omega_s t - r_2 \cos \omega_s t \\ r_1 \cos \omega_s t - r_2 \sin \omega_s t \\ 0 \end{bmatrix}.$$

Further manipulating, for $\sigma > 1$,

$$= \mathbf{e}_i^T r_3 \theta_w \omega_s \begin{bmatrix} \cos \lambda_o t - \cos \omega_s t \\ \sin \lambda_o t - \sin \omega_s t \\ v_3 \end{bmatrix} + \mathbf{e}_i^T \begin{bmatrix} -\,r_1 \omega_s \sin \omega_s t - r_2 \omega_s \cos \omega_s t \\ r_1 \omega_s \cos \omega_s t - r_2 \omega_s \sin \omega_s t \\ 0 \end{bmatrix}.$$

In the last form, the velocity is separated into the component due to normal spin (second component) and perturbations induced by wobble and nutation. Both body a and b are rotating at the same rate, so it is the relative transverse velocity perturbation that is if interest in a separation clearance analysis. Integrating the perturbation

$$d_1(t) = \mathbf{e}_i^T r_3 \theta_w \omega_s \begin{bmatrix} -[\sin\lambda_o t]/\lambda_o + [\sin\omega_s t]/\omega_s \\ [1 - \cos \lambda_o t]/\lambda_o - [1 - \cos \omega_s t]/\omega_s \\ d_3 \end{bmatrix} = \mathbf{e}_i^T r_3 \theta_w \begin{bmatrix} -[\sin\lambda_o t]/\sigma + [\sin\omega_s t] \\ [2\sin^2(\lambda_o t/2)]/\sigma - [2\sin^2(\omega_s t/2)] \\ d_3 \end{bmatrix}. \tag{6.75}$$

In general the motion is complex and must simply be evaluated over time, but if $|\sigma - 1|$, λ_s is small the following approximations obtain,

$$[1 - \cos \lambda_o t]/\lambda_o \approx \frac{1}{\omega_s}[1 - \lambda_s/\omega_s]\{[1 - \cos \omega_s t] - [\cos\lambda_o t - \cos \omega_s t]\}$$

$$= \frac{1}{\omega_s}[1 - \lambda_s/\omega_s]\{[1 - \cos \omega_s t] + 2[\sin(\lambda_s t/2)][\sin(\lambda_o + \omega_s)t/2]\}$$

$$\approx \frac{1}{\omega_s}\{[1 - \cos \omega_s t] + 2[\sin(\lambda_s t/2)][\sin(\lambda_o + \omega_s)t/2]\}$$

$$[\sin\lambda_o t]/\lambda_o \approx \frac{1}{\omega_s}\{[\sin\omega_s t] + 2[\sin(\lambda_s t/2)][\cos(\lambda_o + \omega_s)t/2]\}\ ,$$

and the distance is periodic near spin speed with magnitude increasing relatively slowly comparable to half body nutation period as

$$d_1(t) \approx \mathbf{e}_i^T 2 r_3 \theta_w \sin(\lambda_s t/2) \begin{bmatrix} -\cos[(\lambda_o + \omega_s)t/2] \\ \sin[(\lambda_o + \omega_s)t/2] \\ d_3 \end{bmatrix} \approx \mathbf{e}_i^T 2 r_3 \theta_w \sin(\lambda_s t/2) \begin{bmatrix} -\cos\omega_s t \\ \sin \omega_s t \\ d_3 \end{bmatrix}. \tag{6.76}$$

If, as usually the case, axial clearance occurs in some time δt small compared to body nutation period, the transverse clearance loss may then be bounded by

$$d_1(\delta t) \approx r_3 \theta_w \lambda_s \delta t = r_3 \theta_w (\sigma - 1)\omega_s \delta t = r_3[J_{23}^a/J_T^a]\omega_s \delta t\ ;\ \ \delta t \ll 2\pi/\lambda_s\ . \tag{6.77}$$

6.6 <u>Static Stability and Propellant Migration</u>

As employed here the term static stability refers to stability of the principal axes of inertia with respect to the desired equilibrium spin axis (bearing axis). In a completely rigid body the principal axes are of course fixed to the body. In a flexible body, or a system of bodies with relative motion permitted, the system principal axis orientation will depend on the relative position of the elements. Specifically a rigid body with movable propellant mass and desired principal axis in some nominal geometric orientation with propellant nominally distributed may be stable or otherwise under small imbalance perturbations of the rigid body.

In References 12,13 and 26, the subject of static stability is treated in theory and some specific examples of vehicle geometry are examined. For present purposes, consider a vehicle having four propellant tanks at some fraction fill and a perfectly balanced spinning section (rotor or entire vehicle as applicable to discussion of a particular orbit condition). With the propellant frozen in the perfectly balance equilibrium about the desired spin axis, the spin to transverse inertia ratio is denoted by σ. If a small rotor dynamic imbalance δI is introduced, it produces a principal axis shift

$$\varepsilon_i = \delta I/\{I_T(\sigma - 1)\} \ . \tag{6.78}$$

The vehicle will then spin about the new principal axis and the nominal spin axis will cone about it in the spin frequency motion commonly termed wobble. If the propellant is then unfrozen it will seek a new equilibrium by repositioning within individual tanks and, if unconstrained, by migrating between banks. At this equilibrium the principal axis is displaced from the nominal spin axis by

$$\varepsilon = \delta I/\{I_T[(\sigma - K_p/I_T) - 1]\} = \alpha\varepsilon_i \tag{6.79}$$

where

$$\alpha = 1/[1 - K_p/\{I_T(\sigma - 1)\}] \ . \tag{6.80}$$

α is the wobble amplification factor and K_p is a parameter dependent upon tank geometry and location, propellant density and fraction fill, and total vehicle mass. The spacecraft is said to be statically stable if for arbitrary ε, there exists a δI (or ε_i) such that the resultant principal axis tilt is less than ε. The amplification factor α is plotted qualitatively as Figure 6.5. Here it is observed that $\alpha < 1$ (attenuation) for $\sigma < 1$, and $\alpha > 1$ (amplification) for $\sigma > 1 + K_p/I_T$. In the region $(1, 1 + K_p/I_T)$ the principal axis is unstable, i.e., for small δI propellant will redistribute to produce a large principal axis shift. The equation given for α contains implicit small angle assumptions, so it is not valid in this region. Clearly in the on station configuration when $\sigma < 1$ the effect is beneficial.

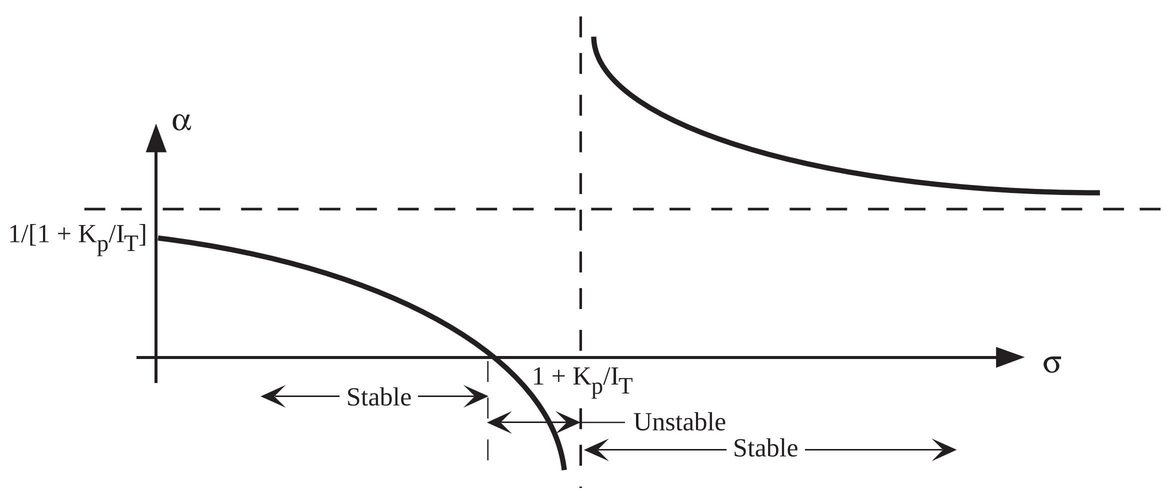


Figure 6.5 Wobble Amplification Factor versus Rigid Body Inertia Ratio.

Propellant Imbalance Amplification on a Single Body Spinning Spacecraft

The derivation herein applies to a single body spinner or to a dual-spin spacecraft with the platform (body not having propellant tanks) despun, by replacing transverse inertias J^s_{ii} with total vehicle transverse inertia. In Ref. 26 some discussions and derivations are given regarding a dual-spin vehicle with both bodies spinning.

Tank Geometry and Propellant Equilibrium Location

For even the simplest tank shapes, location of the free surface and center of mass can be *very* complex. We treat the spherical or equivalent tanks for which the equilibrium free surface is a cylinder about the true spin axis (nearly always the case for any tank shape) and the center of mass is on a radial line in the spin plane passing through the geometric center of the sphere. Several aspects and symbol definitions of this geometry are shown on Figure 6.6.

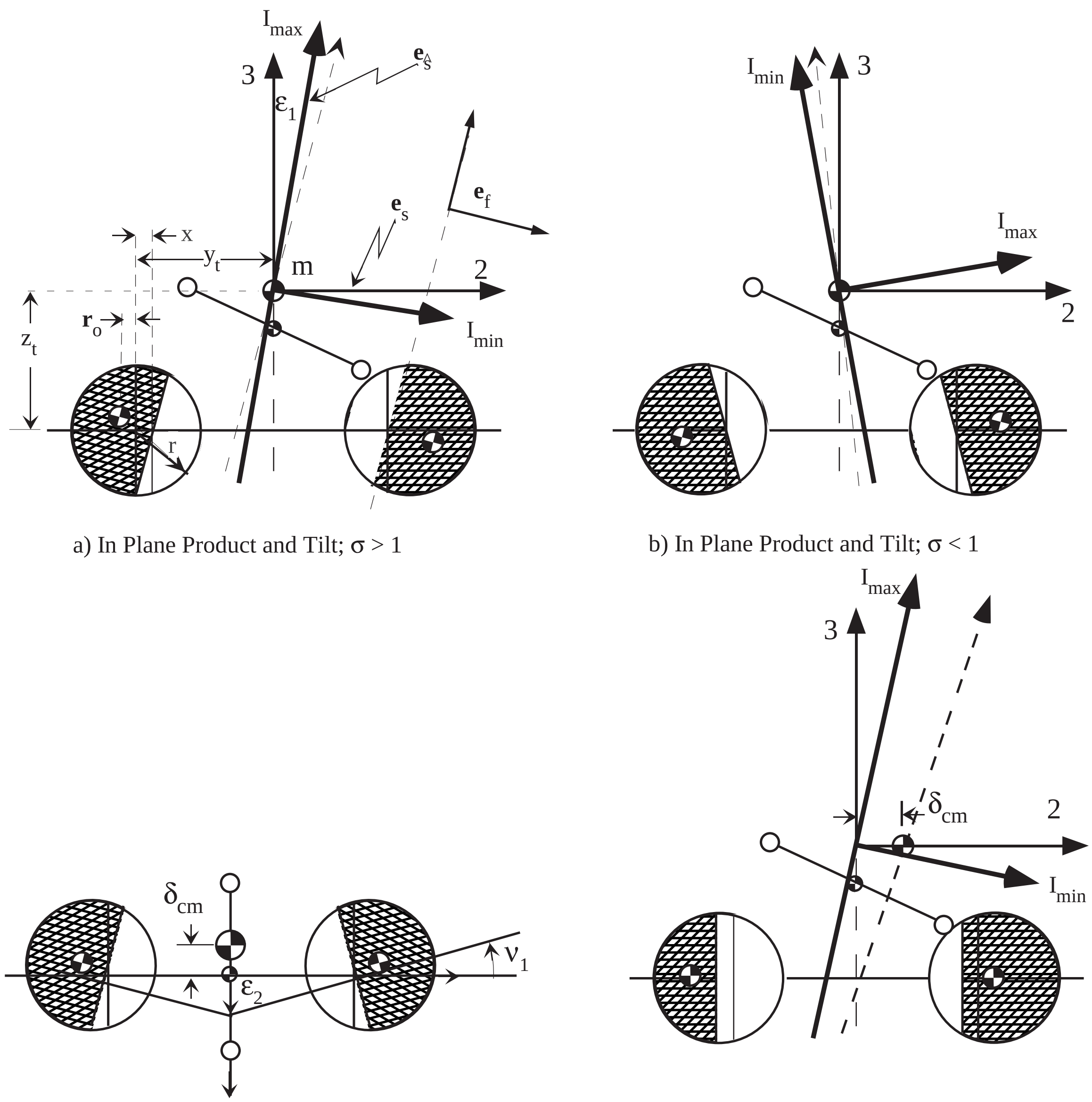


Figure 6.6 Principal Axis of Inertia Repositioning Due to Propellant Tilt and Migration.

Let the body basis be $\mathbf{e}_s$ and assume a principal axis basis, say $\mathbf{e}_{\hat{s}}$, displaced by imbalance tilt angles ε_1, ε_2, such that

$$\mathbf{e}_{\hat{s}} = A_2(\varepsilon_2)A_1(\varepsilon_1)\mathbf{e}_s \ . \tag{6.81}$$

A vector to some tank center of geometric symmetry is

$$\mathbf{r}_t = \mathbf{e}_s^T r_t = \mathbf{e}_{\hat{s}}^T A_2(\varepsilon_2)A_1(\varepsilon_1)r_t = \mathbf{e}_{\hat{s}}^T[x_1, x_2, x_3]^T \tag{6.82}$$

so that

$$\mathbf{r}_t^* = \mathbf{e}_{\hat{s}}^T[x_1, x_2, 0]^T \tag{6.83}$$

is a radial vector in the spin plane passing through the tank center of symmetry, and therefore normal to the fluid free surface. Then

$$\mathbf{u}_t = \mathbf{r}_t^*/|\mathbf{r}_t^*| \tag{6.84}$$

is a unit vector normal to the free surface passing through tank center of symmetry. This vector can be used to establish a new "free surface normal" vector basis $\mathbf{e}_f$ with a convenient orientation to the tank geometry such that cm location and the inertia matrix of propellant in a partially filled tank can be readily computed. For a sphere, or conisphere with sufficient faction fill to be regarded as a sphere, this vector passes through the spherical propellant segment mass center. Mass properties for these geometries are given in Appendix K. Even in this almost trivial case we shall approximate the free surface as planar and normal to $\mathbf{u}_t$, an approximation that improves as the distance of the tank from the spin axis gets large compared to tank radius.

We let $\mathbf{J}_s = \mathbf{e}_s^T J_s \mathbf{e}_s$ denote the rigid body inertia dyadic of the spinning spacecraft with propellant frozen in its equilibrium position about the balanced spin axis where the 3-axis is the spin axis. Introduction of rigid products of inertia in orthogonal planes containing the spin axis results in approximate imbalance wobble angles (solving for rotation angles to null products of inertia in the matrices of Appendix E)

$$\varepsilon_1^o = +(1/2)\mathrm{Tan}^{-1}\{2J_{23}^s/[J_{22}^s - J_{33}^s]\} \approx +J_{23}^s/[J_{22}^s(1 - J_{33}^s/J_{22}^s)] \approx +J_{23}^s/[J_T^s(1-\sigma_2)] \tag{6.85a}$$

$$\varepsilon_2^o = -(1/2)\mathrm{Tan}^{-1}\{2J_{13}^s/[J_{11}^s - J_{33}^s]\} \approx -J_{13}^s/[J_{11}^s(1 - J_{33}^s/J_{11}^s)] \approx -J_{13}^s/[J_T^s(1-\sigma_1)] \tag{6.85b}$$

Propellant Repositioning Within Tanks

Introducing principal axis tilt ε_1 about the 1-axis and carrying out the formality of locating the resulting propellant symmetry vector, we get

$$\mathbf{r}_t = \mathbf{e}_s^T[0, r_t, z_t]^T = \mathbf{e}_{\hat{s}}^T A_1(\varepsilon_1)[0, r_t, z_t]^T = \mathbf{e}_{\hat{s}}^T[0, r_t + z_t\varepsilon_1, z_t - r_t\varepsilon_1]^T \tag{6.86}$$

giving

$$\mathbf{r}_t^* = \mathbf{e}_{\hat{s}}^T[0, r_t + z_t\varepsilon_1, 0]^T \tag{6.87}$$

and

$$\mathbf{u}_t = \mathbf{e}_{\hat{s}}^T[0, 1, 0]^T \ . \tag{6.88}$$

As was clear by inspection, since the tilt rotation axis is normal to the tank radial line in this case the propellant line of symmetry is just the 2-axis of the tilted principal axis system. Hence we need only rotate the propellant inertia back through the tilt angle to determine the perturbed inertia in $\mathbf{e}_s$.

Values for propellant cm and free surface location with respect to the tank center r_o, x, and propellant inertias, J_{ij}^f as a function of tank fill fraction are given in Appendix K. The inertia computed with respect to spacecraft mass center for fuel in one tank rotated by angle ε_1 about the tank center and having cm located at

$$\mathbf{r}_f = \mathbf{r}_t + \mathbf{r}_o = \mathbf{r}_t + \mathbf{e}_s^T A^T(\varepsilon_1)[0, r_o, 0]^T = \mathbf{e}_s^T\{[0, y_t, z_t]^T + r_o[0, \cos\varepsilon_1, \sin\varepsilon_1]^T\} \ , \tag{6.89}$$

as depicted on Figure 6.6a is computed as

$$\mathbf{J}_f(\mathbf{r}_{f_1},\varepsilon_1) = \mathbf{e}_s^T\{A^T(\varepsilon_1)J_fA(\varepsilon_1) - m_f\tilde{r}_{f_1}\tilde{r}_{f_1}\}\mathbf{e}_s = \mathbf{e}_s^T\{A^T(\varepsilon_1)J_fA(\varepsilon_1) - m_f[\tilde{r}_t + [A^T(\varepsilon_1)r_o]^{\sim}][\tilde{r}_t + [A^T(\varepsilon_1)r_o]^{\sim}]\}\mathbf{e}_s$$

$$= \mathbf{e}_s^T\begin{bmatrix} J^f_{11} & -J^f_{12}\cos\varepsilon_1 + J^f_{13}\sin\varepsilon_1 & -J^f_{13}\cos\varepsilon_1 - J^f_{12}\sin\varepsilon_1 \\ -- & J^f_{22} + [J^f_{33} - J^f_{22}]\sin^2\varepsilon_1 + J^f_{23}\sin 2\varepsilon_1 & ------------ \\ -- & -J^f_{23}\cos 2\varepsilon_1 - [(J^f_{33} - J^f_{22})/2]\sin 2\varepsilon_1 & J^f_{33} - [J^f_{33} - J^f_{22}]\sin^2\varepsilon_1 - J^f_{23}\sin 2\varepsilon_1 \end{bmatrix}\mathbf{e}_s$$

$$+ \mathbf{e}_s^T m_f\begin{bmatrix} (y_t + r_o\cos\varepsilon_1)^2 + (z_t + r_o\sin\varepsilon_1)^2 & 0 & 0 \\ 0 & (z_t + r_o\sin\varepsilon_1)^2 & -(y_t + r_o\cos\varepsilon_1)(z_t + r_o\sin\varepsilon_1) \\ 0 & -(y_t + r_o\cos\varepsilon_1)(z_t + r_o\sin\varepsilon_1) & (y_t + r_o\cos\varepsilon_1)^2 \end{bmatrix}\mathbf{e}_s$$

$$\approx \mathbf{e}_s^T A^T(\varepsilon_1)J_fA(\varepsilon_1)\mathbf{e}_s + \mathbf{e}_s^T m_f\begin{bmatrix} (y_t + r_o)^2 + (z_t + r_o\varepsilon_1)^2 & 0 & 0 \\ 0 & (z_t + r_o\varepsilon_1)^2 & -(y_t + r_o)(z_t + r_o\varepsilon_1) \\ 0 & -(y_t + r_o)(z_t + r_o\varepsilon_1) & (y_t + r_o)^2 \end{bmatrix}\mathbf{e}_s\,. \tag{6.90}$$

For reference the untilted inertia is

$$\mathbf{J}_f(0) = \mathbf{e}_s^T[J_f - m_f\tilde{r}_f\tilde{r}_f]\mathbf{e}_s = \mathbf{e}_s^T\begin{bmatrix} J^f_{11} + m_f[z_t^2 + (y_t + r_o)^2] & -J^f_{12} & -J^f_{13} \\ -J^f_{12} & J^f_{22} + m_f z_t^2 & -J^f_{23} - m_f z_t(y_t + r_o) \\ -J^f_{13} & -J^f_{23} - m_f z_t(y_t + r_o) & J^f_{33} + m_f(y_t + r_o)^2 \end{bmatrix}\mathbf{e}_s\,. \tag{6.91}$$

The resultant change in rotor inertia, assuming J^s_{ij}, $i \neq j$, due to the propellant tilt is

$$\delta\mathbf{J}_s(\mathbf{r}_{f_1},\varepsilon_1) = \mathbf{J}_f(\mathbf{r}_{f_1},\varepsilon_1) - \mathbf{J}_f(0) = \tag{6.92a}$$

$$\approx \mathbf{e}_s^T\begin{bmatrix} 2m_f z_t r_o\varepsilon_1 & 0 & 0 \\ 0 & 2m_f z_t r_o\varepsilon_1 & -\{[J^f_{33} - J^f_{22}] + m_f(y_t + r_o)r_o\}\varepsilon_1 \\ 0 & -\{[J^f_{33} - J^f_{22}] + m_f(y_t + r_o)r_o\}\varepsilon_1 & 0 \end{bmatrix}\mathbf{e}_s\,,$$

and for the companion tank on the negative 2-axis

$$\delta\mathbf{J}_s(\mathbf{r}_{f_2},\varepsilon_1) = \mathbf{J}_f(\mathbf{r}_{f_2},\varepsilon_1) - \mathbf{J}_f(0) = \tag{6.92b}$$

$$\approx \mathbf{e}_s^T\begin{bmatrix} -2m_f z_t r_o\varepsilon_1 & 0 & 0 \\ 0 & -2m_f z_t r_o\varepsilon_1 & -\{[J^f_{33} - J^f_{22}] + m_f(y_t + r_o)r_o\}\varepsilon_1 \\ 0 & -\{[J^f_{33} - J^f_{22}] + m_f(y_t + r_o)r_o\}\varepsilon_1 & 0 \end{bmatrix}\mathbf{e}_s\,.$$

Then summing the inertia perturbations

$$\mathbf{J}_s(\varepsilon_1) = \mathbf{J}_s(0) + \sum_i \delta\mathbf{J}_s(\mathbf{r}_{f_i},\varepsilon_1) = \mathbf{J}_s(0) + \delta\mathbf{J}_s(\mathbf{r}_{f_1},\varepsilon_1) + \delta\mathbf{J}_s(\mathbf{r}_{f_2},\varepsilon_1)\,. \tag{6.93}$$

The true amplified principal axis tilt angle ε_1 may be found by solving this implicit equation, e.g., select ε_1 and evaluate $J_s(\varepsilon_1)$, then compute the principal axis tilt of this matrix, and iterate until the selected value and the computed values match. However an approximate value is obtained as follows

$$\delta J^s_{23} = k_p\varepsilon_1 = 2\{[J^f_{33} - J^f_{22}] + m_f(y_t + r_o)r_o\}\varepsilon_1 \to 2m_f(y_t + r_o)r_o\varepsilon_1 \;\;;\; J_f \to 0\,, \tag{6.94a}$$

and

$$\varepsilon_1 = +(1/2)\mathrm{Tan}^{-1}\{2J^s_{23}/[J^s_{22} - J^s_{33}]\} \approx J^s_{23}/[J^s_{22}(1 - J^s_{33}/J^s_{22})] \tag{6.94b}$$

$$= \frac{J^s_{23} + \delta J^s_{23}}{[J^s_T(1 - \sigma_2)]} = \frac{J^s_{23} + k_p\varepsilon_1}{J^s_{22}(1 - \sigma_2)}$$

$$= \frac{J^s_{23}}{J^s_{22}[(1-\sigma_2) - k_p/J^s_{22}]} = \frac{J^s_{23}}{J^s_{22}[1-\sigma_2-\delta\sigma_2]} = \alpha\varepsilon^o_1$$

$$\delta\sigma_2 = k_p/J^s_{22} = 2\{[J^f_{33} - J^f_{22}] + m_f(y_t + r_o)r_o\}\varepsilon_1/J^s_{22} \rightarrow 2m_f(y_t + r_o)r_o\varepsilon_1/J^s_{22} \tag{6.94c}$$

$$\alpha = \frac{1}{1 - \dfrac{k_p}{J^s_{22}(1-\sigma_2)}} \rightarrow \frac{1}{1 - \dfrac{2m_f(y_t + r_o)r_o}{J^s_{22}(1-\sigma_2)}} \ . \tag{6.94d}$$

For a tilt about the 2-axis induced by rigid product of inertia J^s_{13} we derive in analogous fashion

$$\mathbf{r}_t = \mathbf{e}^T_s[0, y_t, z_t]^T = \mathbf{e}^T_{\hat{s}}A_2(\varepsilon_2)[0, y_t, z_t]^T = \mathbf{e}^T_{\hat{s}}[-z_t\varepsilon_2, y_t, z_t]^T \tag{6.95}$$

leading to

$$\mathbf{r}^*_t = \mathbf{e}^T_{\hat{s}}[-z_t\varepsilon_2, y_t, 0]^T \tag{6.96}$$

and the propellant symmetry vector

$$\mathbf{u}_t = \frac{\mathbf{e}^T_{\hat{s}}[-z_t\varepsilon_2, y_t, 0]^T}{\sqrt{y_t^2 + (z_t\varepsilon_2)^2}} \approx \mathbf{e}^T_{\hat{s}}[-(z_t/y_t)\varepsilon_2, 1, 0]^T \ , \tag{6.97}$$

which indicates that the propellant free surface basis $\mathbf{e}_f$ is rotated by small angle $\nu_3 = -(z_t/y_t)\varepsilon_2$ about the 3-axis of $\mathbf{e}_{\hat{s}}$. To express the perturbed propellant inertia in $\mathbf{e}_s$ we must first rotate propellant cm and inertia parameters $\mathbf{r}_o$ and $\mathbf{J}$ through ν_3 about the 3-axis, and then by ε_2 about the 2-axis as

$$\mathbf{J}_f(\nu_3, \varepsilon_2) = \mathbf{e}^T_s\{A^T(\varepsilon_2)A^T(\nu_3)JA(\nu_3)A(\varepsilon_3) - m_f\tilde{r}_f\tilde{r}_f]\}\mathbf{e}_s \tag{6.98}$$

$$= \mathbf{e}^T_s\{A^T(\varepsilon_2)A^T(\nu_3)JA(\nu_3)A(\varepsilon_3) - m_f[\tilde{r}_t + [A^T(\varepsilon_2)A^T(\nu_3)r_o]^{\sim}\ ][\tilde{r}_t + [A^T(\varepsilon_2)A^T(\nu_3)r_o]^{\sim}\ ] \ . \}\mathbf{e}_s$$

Forming the vehicle cm to propellant cm vector for this case

$$\mathbf{r}_f = \mathbf{r}_t + \mathbf{r}_o = \mathbf{r}_t + \mathbf{e}^T_sA^T(\varepsilon_2)A^T(\nu_3)[0, r_o, 0]^T \tag{6.99}$$

$$= \mathbf{e}^T_s\{[0, y_t, z_t]^T + r_o[-\cos\varepsilon_2\sin\nu_3, \cos\nu_3, \sin\varepsilon_2\sin\nu_3]^T\} \approx \mathbf{e}^T_s[-r_o\nu_3, y_t + r_o, z_t]^T\}$$

$$= \mathbf{e}^T_s[r_o(z_t/y_t)\varepsilon_2, y_t + r_o, z_t]^T\} \ .$$

The propellant inertia becomes

$$\mathbf{J}_f(\nu_3, \varepsilon_2) \approx \mathbf{e}^T_sA^T(\varepsilon_2)A^T(\nu_3)J_fA(\nu_3)A(\varepsilon_2)\mathbf{e}_s + \mathbf{e}^T_sm_f\begin{bmatrix} (y_t + r_o)^2 + z_t^2 & -r_oz_t\varepsilon_2 & -r_o(z_t/y_t)z_t\varepsilon_2 \\ -r_oz_t\varepsilon_2 & z_t^2 & -(y_t + r_o)z_t \\ -r_o(z_t/y_t)z_t\varepsilon_2 & -(y_t + r_o)z_t & (y_t + r_o)^2 \end{bmatrix}\mathbf{e}_s \ . \tag{6.100}$$

Again taking two symmetrically positioned tanks and expanding only terms pertaining to the $J_f \rightarrow 0$ limit, the total perturbation is

$$\mathbf{J}_s(\nu_3, \varepsilon_2) = \mathbf{J}_s(0) + \delta\mathbf{J}_s(\nu_3, \varepsilon_2) + \delta\mathbf{J}_s(-\nu_3, \varepsilon_2) \tag{6.101}$$

$$= \mathbf{J}_s(0) + \ + \mathbf{e}^T_s2m_f\begin{bmatrix} 0 & 0 & -r_o(z_t/y_t)z_t\varepsilon_2 \\ 0 & 0 & 0 \\ -r_o(z_t/y_t)z_t\varepsilon_2 & 0 & 0 \end{bmatrix}\mathbf{e}_s \ .$$

In addition the shifted propellant induces a small vehicle mass center shift as

$$\delta_{cm} = (m_f/m)r_o\nu_3 = -(m_f/m)r_o(z_t/y_t)\varepsilon_2 \tag{6.102}$$

resulting in a reduction of the tilt induced product by

$$\delta J_{13}^{cm} = -\, m_f(m_f/m)(r_o/y_t)z_t^2\varepsilon_2 \tag{6.103}$$

and producing a residual product for two tanks

$$\delta J_{13}^{s} = k_p\varepsilon_2 = 2m_f(1 - m_f/m)(r_o/y_t)z_t^2\varepsilon_2 \tag{6.104a}$$

$$\varepsilon_2 = -\,(1/2)\mathrm{Tan}^{-1}\{2J_{13}^s/[J_{11}^s - J_{33}^s]\} \approx -\,J_{13}^s/[J_{11}^s(1 - J_{33}^s/J_{11}^s)] \approx -\,J_{13}^s/[J_{11}^s(1 - \sigma_1)] \tag{6.104b}$$

$$= \frac{-J_{13}^s - \delta J_{13}^s}{J_{11}^s(1 - \sigma_1)} = \frac{-J_{13}^s + k_p\varepsilon_2}{J_{11}^s(1 - \sigma_1)}$$

$$= \frac{-J_{13}^s}{J_{11}^s[(1 - \sigma_1) - k_p/J_{11}^s]} = \frac{-J_{13}^s}{J_{11}^s[1 - \sigma_1 - \delta\sigma_1]} = \alpha\varepsilon_2^o$$

$$\delta\sigma_1 = k_p/J_{11}^s = 2m_f(1 - m_f/m)(r_o/y_t)z_t^2/J_{11}^s \tag{6.104c}$$

$$\alpha = \frac{1}{1 - \dfrac{k_p}{J_{11}^s(1 - \sigma_1)}} = \frac{1}{1 - \dfrac{2m_f(1 - m_f/m)(r_o/y_t)z_t^2}{J_{11}^s(1 - \sigma_1)}}\ . \tag{6.104d}$$

Migration of Propellant Between Tanks

Forming the position vector to the respective free surfaces in the tilted basis $\mathbf{e}_{\hat{s}}$,

$$\mathbf{r}_f^a = \mathbf{e}_s^T[0,\ y_t + x_a - \delta_{cm},\ z_t]^T = \mathbf{e}_{\hat{s}}^T A_2(\varepsilon_2)A_1(\varepsilon_1)[0,\ y_t + x_a - \delta_{cm},\ z_t]^T \tag{6.105b}$$

$$= \mathbf{e}_{\hat{s}}^T[-z_t\varepsilon_2,\ y_t + x_a - \delta_{cm} + z_t\varepsilon_1,\ z_t - (y_t + x_a - \delta_{cm})\varepsilon_1]^T$$

$$\mathbf{r}_f^b = \mathbf{e}_s^T[0,\ -\,y_t - x_b - \delta_{cm},\ z_t]^T = \mathbf{e}_{\hat{s}}^T A_2(\varepsilon_2)A_1(\varepsilon_1)[0,\ -\,y_t - x_b - \delta_{cm},\ z_t]^T\ . \tag{6.105b}$$

$$= \mathbf{e}_{\hat{s}}^T[-z_t\varepsilon_2,\ -\,y_t - x_b - \delta_{cm} + z_t\varepsilon_1,\ z_t + (y_t + x_b + \delta_{cm})\varepsilon_1]^T$$

The mass transfer is

$$\delta m \approx \pi\rho(r^2 - x_o^2)\delta x\ , \tag{6.106}$$

with accompanying vehicle cm shift

$$\delta_{cm} \approx 2(\delta m/m)(y_t + x_o) = (2/m)\pi\rho(r^2 - x_o^2)\delta x(y_t + x_o)\ . \tag{6.107}$$

Equating spin plane components of distance to the free surface from (6.101) yields

$$(x_b - x_a)/2 = \delta x = z_t\varepsilon_1 - \delta_{cm} = z_t\varepsilon_1 - (2/m)\pi\rho(r^2 - x_o^2)\delta x(y_t + x_o) \tag{6.108}$$

so that solving for δx yields

$$\delta x = \frac{z_t\varepsilon_1}{1 + (2/m)\pi\rho(r^2 - x_o^2)(y_t + x_o)}$$

while

$$\delta m \approx \frac{\pi\rho(r^2 - x_o^2)z_t\varepsilon_1}{1 + (2/m)\pi\rho(r^2 - x_o^2)(y_t + x_o)}\ , \tag{6.109}$$

and finally

$$\delta_{cm} \approx \frac{(2/m)\pi\rho(r^2 - x_o^2)(y_t + x_o)z_t\varepsilon_1}{1 + (2/m)\pi\rho(r^2 - x_o^2)(y_t + x_o)}\ . \tag{6.110}$$

The product of inertia induced by propellant migration is

$$\delta J^s_{23} = 2\delta m(y_t + x_o)z_t = \frac{2\pi\rho(r^2 - x_o^2)(y_t + x_o)z_t^2\varepsilon_1}{1 + (2/m)\pi\rho(r^2 - x_o^2)(y_t + x_o)} = k_p\varepsilon_1 \ . \tag{6.111a}$$

Observe that migration does not alter the moments of inertia. Solving for the principal axis tilt amplified by migration

$$\varepsilon_1 = +(1/2)\mathrm{Tan}^{-1}\{2J^s_{23}/[J^s_{22} - J^s_{33}]\} \approx J^s_{23}/[J^s_{22}(1 - J^s_{33}/J^s_{22})] \approx J^s_{23}/[J^s_{22}(1 - \sigma_2)] \tag{6.111b}$$

$$= \frac{J^s_{23} + \delta J^s_{23}}{[J^s_{22}(1 - \sigma_2)]} = \frac{J^s_{23} + k_p\varepsilon_1}{[J^s_{22}(1 - \sigma_2)]}$$

$$= \frac{J^s_{23}}{[J^s_{22}(1 - \sigma_2 - k_p/J^s_{22})]} = \frac{J^s_{23}}{[J^s_{22}(1 - \sigma_2 - \delta\sigma_2)]} = \alpha\varepsilon_1^o$$

$$\delta\sigma_2 = k_p/J^s_{22} = \frac{2\pi\rho(r^2 - x_o^2)(y_t + x_o)z_t^2}{J^s_{22}[1 + (2/m)\rho\pi(r^2 - x_o^2)(y_t + x_o)]} \tag{6.111c}$$

$$\alpha = \frac{1}{1 - \dfrac{k_p}{J^s_{22}(1 - \sigma_2)}} \ . \tag{6.111d}$$

Summary of Propellant Wobble Amplification Effect

For a pair of two spherical tanks located on a diameter of the spacecraft the total wobble amplification effect due to both migration and repositioning of propellant can be summarized by collecting the results of (6.93, 99, and 107) as follows.

Amplification In Tank Plane

$$k_p = 2[J^f_{33} - J^f_{22}] + 2m_f(y_t + r_o)r_o + \frac{2\pi\rho(r^2 - x_o^2)(y_t + x_o)z_t^2}{1 + (2/m)\rho\pi(r^2 - x_o^2)(y_t + x_o)} \tag{6.112a}$$

$$\rightarrow 2m_f(y_t + r_o)r_o + \frac{2\pi\rho(r^2 - x_o^2)(y_t + x_o)z_t^2}{1 + (2/m)\rho\pi(r^2 - x_o^2)(y_t + x_o)} \ \ ; J \rightarrow 0$$

$$\varepsilon_1 = +\frac{1}{2}\mathrm{Tan}^{-1}\left\{\frac{2J^s_{23}}{[J^s_{22} - J^s_{33}]}\right\} \approx \frac{J^s_{23}}{[J^s_{22}(1 - J^s_{33}/J^s_{22})]} \approx \frac{J^s_{23}}{[J^s_T(1 - \sigma_2)]} \approx \frac{J^s_{23} + k_p\varepsilon_1}{J^s_{22}(1 - \sigma_2)} \tag{6.112b}$$

$$\approx \frac{J^s_{23}}{J^s_{22}[(1 - \sigma_2) - k_p/J^s_{22}]} = \frac{J^s_{23}}{J^s_{22}[1 - \sigma_2 - \delta\sigma_2]} = \alpha\varepsilon_1(0) = \alpha\varepsilon_1^0$$

$$\delta\sigma_2 = k_p/J^s_{22} \approx k_p/J^s_{22} \ \ ; \alpha = \frac{1}{1 - k_p/[J^s_{22}(1 - \sigma_2)]} \tag{6.112c}$$

Amplification Normal to Tank Plane

$$k_p = 2(1 - m_f/m)m_f(r_o/y_t)z_t^2 \tag{6.113a}$$

$$\varepsilon_2 = -\frac{1}{2}\mathrm{Tan}^{-1}\left\{\frac{2J^s_{13}}{[J^s_{11} - J^s_{33}]}\right\} \approx \frac{-J^s_{13}}{[J^s_{11}(1 - J^s_{33}/J^s_{11})]} = \frac{-J^s_{13}}{[J^s_{11}(1 - \sigma_1)]} = \frac{-J^s_{13} + k_p\varepsilon_2}{J^s_{11}(1 - \sigma_1)} \tag{6.113b}$$

$$= \frac{-J^s_{13}}{J^s_{11}[(1 - \sigma_1) - k_p/J^s_{11}]} = \frac{-J^s_{13}}{J^s_{11}[1 - \sigma_1 - \delta\sigma_1]} = \alpha\varepsilon_2(0) = \alpha\varepsilon_2^0$$

$$\delta\sigma_2 = k_p/J^s_{11} \approx k_p/J^s_{11} \ \ ; \alpha = \frac{1}{1 - k_p/[J^s_{11}(1 - \sigma_1)]} \ . \tag{6.113c}$$

6.7 Mass Property Perturbation Due to Propellant Repositioning Under Vehicle Acceleration

Under axial acceleration the propellant in partially filled tanks will shift to alter the vehicle mass properties. A simple case of spherical tanks is illustrated on Figure 6.7.

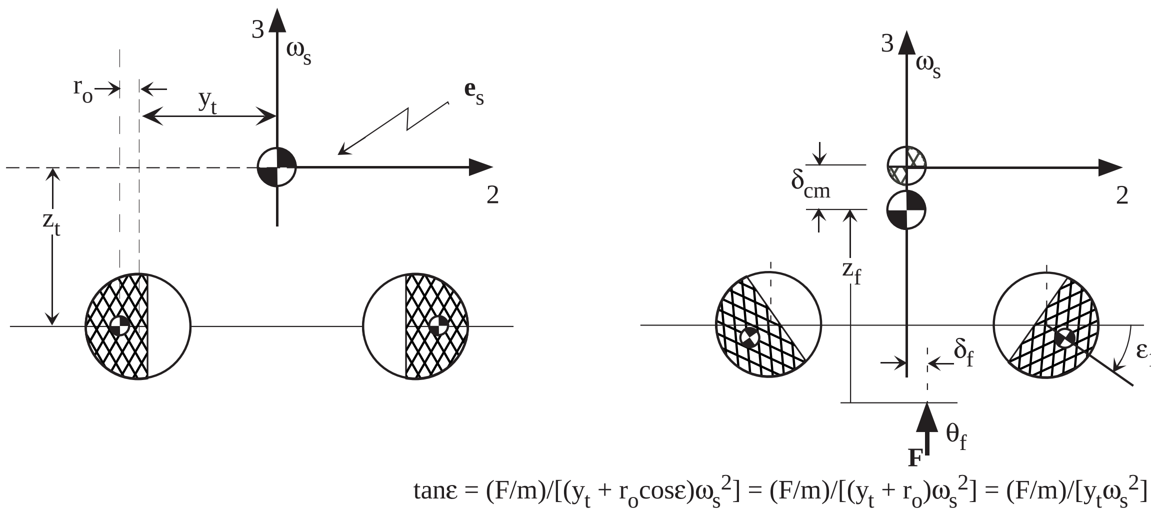


Figure 6.7 Propellant Mass Shift Induced by Axial Acceleration.

In the simple spherical tank case the propellant mass and cm rotate about the tank center through an angle whose tangent is the ratio of axial acceleration F/m to centrifugal acceleration of the spin field given as

$$\tan\varepsilon_1 = (F/m)/[(y_t + r_o\cos\varepsilon_1)\omega_s^2] \approx (F/m)/[(y_t + r_o)\omega_s^2]\ . \qquad (6.114)$$

The inertia matrix expansion given for a single tank in (6.90) is adequate for calculating the change in inertia induced by propellant tilt. Note that the tilt angle is not necessarily small so the full expansion is retained.

$$\delta\mathbf{J}_s(\mathbf{r}_{f_1},\varepsilon_1) = \delta\mathbf{J}_f(\mathbf{r}_{f_1},\varepsilon_1) = \mathbf{J}_f(\mathbf{r}_{f_1},\varepsilon_1) - \mathbf{J}_f(0) \qquad (6.115)$$

$$= \mathbf{e}_s^T\begin{bmatrix} 0 & -J_{12}(\cos\varepsilon_1 - 1) + J_{13}\sin\varepsilon_1 & -J_{13}(\cos\varepsilon_1 - 1) - J_{12}\sin\varepsilon_1 \\ -- & [J_{33} - J_{22}]\sin^2\varepsilon_1 + J_{23}\sin 2\varepsilon_1 & ------------ \\ -- & -J_{23}(\cos 2\varepsilon_1 - 1) - [(J_{33} - J_{22})/2]\sin 2\varepsilon_1 & -[J_{33} - J_{22}]\sin^2\varepsilon_1 - J_{23}\sin 2\varepsilon_1 \end{bmatrix}\mathbf{e}_s$$

$$+ \mathbf{e}_s^T m_f\begin{bmatrix} 2y_t r_o(\cos\varepsilon_1 - 1) + r_o^2(\cos^2\varepsilon_1 - 1) + (2z_t + r_o\sin\varepsilon_1)r_o\sin\varepsilon_1 & 0 & 0 \\ 0 & (2z_t + r_o\sin\varepsilon_1)r_o\sin\varepsilon_1 & -[z_t r_o(\cos\varepsilon_1 - 1) + y_t r_o\sin\varepsilon_1 + r_o^2\sin\varepsilon_1\cos\varepsilon_1] \\ 0 & -[z_t r_o(\cos\varepsilon_1 - 1) + y_t r_o\sin\varepsilon_1 + r_o^2\sin\varepsilon_1\cos\varepsilon_1] & 2y_t r_o(\cos\varepsilon_1 - 1) + r_o^2(\cos^2\varepsilon_1 - 1) \end{bmatrix}\mathbf{e}_s\ .$$

The total inertia perturbation due to two diagonally located tanks as depicted in Figure 6.7, omitting products of inertia which will vanish, is

$$\delta\mathbf{J}_s = \delta\mathbf{J}_f(\mathbf{r}_{f_1},\varepsilon_1) + \delta\mathbf{J}_f(\mathbf{r}_{f_2},-\varepsilon_1) \qquad (6.116)$$

$$\delta J_{11} = 2m_f[2y_t r_o(\cos\varepsilon_1 - 1) + r_o^2(\cos^2\varepsilon_1 - 1) + (2z_t + r_o\sin\varepsilon_1)r_o\sin\varepsilon_1] \approx 4m_f r_o[y_t(\cos\varepsilon_1 - 1) + z_t\sin\varepsilon_1]$$

$$\delta J_{22} = 2[J_{33} - J_{22}]\sin^2\varepsilon_1 + 2m_f[(2z_t + r_o\sin\varepsilon_1)r_o\sin\varepsilon_1] \approx 2[J_{33} - J_{22}]\sin^2\varepsilon_1 + 4m_f r_o z_t\sin\varepsilon_1$$

$$\delta J_{33} = -2[J_{33} - J_{22}]\sin^2\varepsilon_1 + 2m_f[2y_t r_o(\cos\varepsilon_1 - 1) + r_o^2(\cos^2\varepsilon_1 - 1)] \approx -2[J_{33} - J_{22}]\sin^2\varepsilon_1 + 4m_f r_o y_t(\cos\varepsilon_1 - 1)\ .$$

The self inertias, J_{ii}, are also typically small and the effect is of most interest when inertia ratio $\sigma \approx 1$, leading to the following approximate relations

$$\partial\sigma_1 = \frac{\partial J_{33}}{J_{11}} - \frac{J_{33}}{J_{11}}\frac{\partial J_{11}}{J_{11}} = \frac{\delta J_{33}}{J_{11}} - \sigma_1\frac{\delta J_{11}}{J_{11}} \approx \frac{-4m_f r_o[z_t \sin\varepsilon_1]}{J_{11}} \tag{6.117a}$$

$$\partial\sigma_2 = \frac{\partial J_{33}}{J_{22}} - \frac{J_{33}}{J_{22}}\frac{\partial J_{22}}{J_{22}} = \frac{\delta J_{33}}{J_{22}} - \sigma_2\frac{\delta J_{22}}{J_{22}} \approx \frac{-4m_f r_o[z_t \sin\varepsilon_1 - y_t(\cos\varepsilon_1 - 1)]}{J_{22}}\ . \tag{6.117b}$$

The cm shift depicted on Figure 6.7 is

$$\delta_{cm} = -\,2(m_f/m)r_o \sin\varepsilon_1$$

Denote the initial mass center 3-axis station as

$$z_{cm} = \sum_i (m_i/m)z_i$$

then the change in transverse inertia due to the cm migration is

$$\delta J_T = \sum_i [(z_i - z_{cm} + \delta_{cm})^2 - (z_i - z_{cm})^2]m_i = \delta_{cm}\sum_i [2(z_i - z_{cm}) + \delta_{cm}]m_i \tag{6.118}$$

$$= \delta_{cm}^2 \sum_i m_i = m\delta_{cm}^2 = 4(m_f/m)m_f[r_o \sin\varepsilon_1]^2 \ll 4m_f r_o[z_t \sin\varepsilon_1]\ .$$

Inspection of the inequality in comparison to (6.117) indicates that the cm shift term is generally negligible.

6.8 Propellant Transport

When propellant is expended it is frequently transported to a different spin radius before expulsion through a thruster. In this transport the vehicle (including propellant) spin inertia is altered, inducing a change in spin speed. An expression for the combined thruster torque impulse imparted while mass and inertia are changing due to mass expulsion and transport of propellant to the point of expulsion is, considering the spin axis component only

$$T_3 = \frac{d\{I_{33}\cdot\omega_s\}}{dt} = \dot{I}_{33}\omega_s + I_{33}\dot{\omega}_s = Fr_j\alpha_j \tag{6.119}$$

where r_j, α_j are radial moment arm and alignment angle of the thruster. Approximating $\dot{I}_{33}$ as constant which covers many practical cases, a time invariant linear differential equation results giving transform

$$[\dot{I}_{33} + I_{33}s]\omega_s = I_{33}\omega_s(0) + Fr_j\alpha_j/s \tag{6.120}$$

$$\omega_s = \frac{\omega_s(0)}{[s + \dot{I}_{33}/I_{33}]} + \frac{Fr_j\alpha_j/I_{33}}{s[s + \dot{I}_{33}/I_{33}]} = \frac{\omega_s(0)}{[s + \dot{I}_{33}/I_{33}]} + Fr_j\alpha_j/\dot{I}_{33}\left[\frac{1}{s} - \frac{1}{[s + \dot{I}_{33}/I_{33}]}\right]$$

whose solution is

$$\omega_s(t) = \omega_s(0)e^{-t\dot{I}_{33}/I_{33}} + Fr_j\alpha_j/\dot{I}_{33}[1 - e^{-t\dot{I}_{33}/I_{33}}] \tag{6.121}$$

$$\omega_s(t) - \omega_s(0) = \delta\omega_s = -\,[\omega_s(0) - Fr_j\alpha_j/\dot{I}_{33}][1 - e^{-t\dot{I}_{33}/I_{33}}] \approx -\,[\omega_s(0) - Fr_j\alpha_j/\dot{I}_{33}][t\dot{I}_{33}/I_{33}]\ .$$

Then expressing mass flow rate and the inertia derivative as

$$\frac{dm}{dt} = \frac{d}{dt}\left[\frac{Ft}{gI_{sp}}\right] = \left[\frac{F}{gI_{sp}}\right] \tag{6.122}$$

$$t = [gI_{sp}/F][m_f - m_i] = [gI_{sp}/F]\delta m \tag{6.123}$$

$$\frac{dI_{33}}{dt} = \dot{I}_{33} = (r_j^2 - r_p^2)\frac{dm}{dt} = (r_j^2 - r_p^2)\frac{F}{gI_{sp}}\ . \tag{6.124}$$

In this expression m_i, m_f are initial and final propellant mass on board assumed equal (or average) at radius of r_p, r_j is the radius of propellant expulsion, $T_3 = F\alpha_j r_j$ is thruster spin torque, and J_{33}, $I_{33} = J_{33} + m_f r_p^2$ are vehicle spin inertias without and with propellant respectively. The spin rate change can be expressed in terms of mass change and independent of time as

$$\delta\omega_s \approx -[\omega_s(0) - Fr_j\alpha_j/\dot{I}_{33}]\left[\frac{[m_f - m_i](r_j^2 - r_p^2)}{I_{33}}\right] = -[\omega_s(0) - gI_{sp}r_j\alpha_j/(r_j^2 - r_p^2)]\delta m(r_j^2 - r_p^2)/I_{33}$$

$$= -\omega_s(0)\delta m(r_j^2 - r_p^2)/I_{33} + gI_{sp}r_j\alpha_j\delta m/I_{33} = -\omega_s(0)\delta m(r_j^2 - r_p^2)/I_{33} + Fr_j\alpha_j t/I_{33}\ . \qquad (6.125)$$

Another approximate approach for integration of (6.119) follows by eliminating the time variable below. Write

$$\dot{I}_{33}\omega_s + I_{33}\dot{\omega}_s = \omega_s(r_j^2 - r_p^2)\frac{dm}{dt} + (J_{33} + mr_p^2)\frac{d\omega_s}{dt} = Fr_j\alpha_j = gI_{sp}r_j\alpha_j\frac{dm}{dt} \qquad (6.126)$$

then rearrange and integrate as

$$\frac{d\omega_s}{\omega_s(r_j^2 - r_p^2) - gI_{sp}r_j\alpha_j} = \frac{dm}{J_{33} + mr_p^2} \qquad (6.127)$$

$$\ln\left[\frac{\omega_f - \dfrac{gI_{sp}r_j\alpha_j}{(r_j^2 - r_p^2)}}{\omega_i - \dfrac{gI_{sp}r_j\alpha_j}{(r_j^2 - r_p^2)}}\right] = \frac{(r_j^2 - r_p^2)}{r_p^2}\ln\left[\frac{J_{33} + m_f r_p^2}{J_{33} + m_i r_p^2}\right]$$

yielding finally

$$\delta\omega_s = \omega_f - \omega_i = [\omega_i - gI_{sp}r_j\alpha_j/(r_j^2 - r_p^2)]\left\{\left[\frac{J_{33} + m_f r_p^2}{J_{33} + m_i r_p^2}\right]^{[(r_j/r_p)^2 - 1]} - 1\right\} \qquad (6.128)$$

$$= [\omega_i - gI_{sp}r_j\alpha_j/(r_j^2 - r_p^2)]\left\{\left[\frac{I_{33}}{I_{33} + \delta m r_p^2}\right]^{[(r_j/r_p)^2 - 1]} - 1\right\}$$

$$\approx [\omega_i - gI_{sp}r_j\alpha_j/(r_j^2 - r_p^2)]\left[\frac{-\delta m(r_j^2 - r_p^2)}{I_{33}}\right];\quad \delta m r_p^2/I_{33} \ll 1$$

$$= -\omega_i\delta m(r_j^2 - r_p^2)/I_{33} - g\delta m I_{sp}r_j\alpha_j/I_{33} = -\omega_i\delta m(r_j^2 - r_p^2)/I_{33} + T_3\delta t/I_{33}\ .$$

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